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An Analogue of the Coleman–Mandula Theorem for Quantum Field Theory in Curved Spacetimes

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Dedicated to the memory of Rudolf Haag

Abstract: The Coleman–Mandula (CM) theorem states that the Poincaré and internal symmetries of a Minkowski spacetime quantum field theory cannot combine nontrivially in an extended symmetry group. We establish an analogous result for quantum field theory in curved spacetimes, assuming local covariance, the timeslice property, a local dynamical form of Lorentz invariance, and additivity. Unlike the CM theorem, our result is valid in dimensions $n \geq 2$ and for free or interacting theories. It is formulated for theories defined on a category of all globally hyperbolic spacetimes equipped with a global coframe, on which the restricted Lorentz group acts, and makes use of a general analysis of symmetries induced by the action of a group G on the category of spacetimes. Such symmetries are shown to be canonically associated with a cohomology class in the second degree nonabelian cohomology of G with coefficients in the global gauge group of the theory. Our main result proves that the cohomology class is trivial if G is the universal cover $\mathcal{S}$ of the restricted Lorentz group. Among other consequences, it follows that the extended symmetry group is a direct product of the global gauge group and $\mathcal{S}$, all fields transform in multiplets of $\mathcal{S}$, fields of different spin do not mix under the extended group, and the occurrence of noninteger spin is controlled by the centre of the global gauge group. The general analysis is also applied to rigid scale covariance.

1. Introduction

In the issue of Communications in Mathematical Physics dedicated to Rudolf Haag's 80th birthday, Brunetti, Fredenhagen and Verch [4] introduced *locally covariant quantum field theory*, a formulation of QFT in curved spacetimes that is a far-reaching generalization of Haag's framework of local quantum physics [21] (also called algebraic QFT). Locally covariant QFTs are expressed as functors from a category of spacetimes BkGrnd to a category of physical systems Phys. The morphisms of BkGrnd correspond to embeddings of one spacetime as a subspacetime of another, while the morphisms of Phys correspond to embeddings of one physical system as a subsystem of another. A functor $\mathcal{A}$: $\mathsf{BkGrnd} \to \mathsf{Phys}$ therefore associates a physical system to every spacetime and also specifies how each spacetime embedding gives an embedding of these physical systems. Thus, $\mathcal{A}$ defines the theory on all spacetimes and incorporates the principle of locality from the start. Locally covariant QFT has proved to be a fruitful framework for the general analysis of QFT in curved spacetime and has allowed various structural results or properties of flat spacetime QFT to be transferred to curved spacetimes (see [20] for a review). Examples include the spin-statistics connection [41], the analysis of superselection sectors [5], Reeh–Schlieder and split properties [16,38], punctured Haag duality [37], and modular nuclearity [28]; one can also discuss the question of whether a theory represents the same physics in all spacetimes [19]. These ideas also play a central role in constructions of perturbative QFT in curved spacetimes [24,25,36].

The aim of this paper is to formulate and prove an analogue of the Coleman–Mandula (CM) Theorem [7] for locally covariant QFT on general parallelizable globally hyperbolic spacetimes of dimension $n \geq 2$. The CM theorem originated as part of an intensive effort in the 1960's to understand whether the internal and Poincaré symmetries of a QFT in Minkowski space could be combined ('mixed') in a larger symmetry group other than as a direct product. These investigations led to a series of no-go theorems of increasing scope based on group theoretic grounds [27,30,32] or, as with the CM result itself (and its generalizations to dimensions $n > 4$ [34]), on dynamical considerations centred on the S-matrix. Later, supersymmetry offered a potential loophole to these results, because fermionic charges interchange bosonic and fermionic fields and therefore also change spin. One of Haag's most highly cited papers was his joint work with Łopuszański and Sohnius [22], in which they showed that the structure of the super Lie algebra in theories obeying certain basic requirements is tightly constrained: in the massive case, for example, internal and Poincaré symmetries commute and the fermionic charges must commute with translations and transform as rank-1 spinors under the Lorentz group.

The CM theorem concerns a particular spacetime of high symmetry. For a generic spacetime with trivial isometry group, it is obvious that the internal and geometric symmetries combine as a direct product, and one might think that the CM theorem has nothing to say except for spacetimes of high symmetry (see [8] for a recent CM analogue in de Sitter spacetime). However, the viewpoint of locally covariant QFT suggests a different approach. Rather than focus on particular spacetimes, we will prove a result (Theorem 11) that applies to the theory across *all* spacetimes, and is expressed in terms of properties of the corresponding functor. We caution that our result should not be viewed as a direct generalization of the CM theorem, but nonetheless maintain that it is a natural analogue thereof in the context of locally covariant QFT. Theorem 11 shares with the CM theorem an emphasis on dynamics, but its method of proof is quite different, and the statement differs from the CM theorem in important respects: notably, it is valid in all spacetime dimensions $n \geq 2$ and it is not assumed that the QFT in question is interacting—whereas there are well-known free theories and two-dimensional models that evade CM. We comment more on these points below after first explaining the main ideas of our approach.

It is necessary to recall two ways in which symmetry can be exhibited by a locally covariant theory $\mathcal{A}$: $\mathsf{BkGrnd} \to \mathsf{Phys}$. First, the *spacetime symmetries* of a spacetime $\boldsymbol{M}$ are just the automorphisms $\psi : \boldsymbol{M} \to \boldsymbol{M}$ in BkGrnd. Any such automorphism is mapped automatically to an automorphism $\mathcal{A}(\psi)$ of the physical system $\mathcal{A}(\boldsymbol{M})$ of the theory on $\boldsymbol{M}$, and for two such symmetries one has $\mathcal{A}(\psi) \circ \mathcal{A}(\varphi) = \mathcal{A}(\psi \circ \varphi)$ by functoriality. In this way, the (generically trivial) group $\mathrm{Aut}(\boldsymbol{M})$ of spacetime symmetries of $\boldsymbol{M}$ is represented in the automorphism group of $\mathcal{A}(\boldsymbol{M})$. Second, the *internal symmetries* of the theory have a natural description. Any functor $\mathcal{A}$ has an associated group, $\mathrm{Aut}(\mathcal{A})$, consisting of all natural isomorphisms of $\mathcal{A}$ to itself. In locally covariant QFT, $\mathrm{Aut}(\mathcal{A})$ is the global gauge group of the theory [15]. It follows from the definition that internal symmetries commute with spacetime symmetries. For this reason we will focus on their combination with the Lorentz group.

In order to give the Lorentz group some purchase in curved spacetimes, Theorem 11 is formulated for locally covariant theories defined on $\mathsf{BkGrnd} = \mathsf{FLoc}$, the category of all n-dimensional globally hyperbolic spacetimes equipped with a global coframe $e = (e^\mu)_{\mu=0}^{n-1}$ for the metric $g = \eta_{\mu\nu}e^\mu \otimes e^\nu$. Among other requirements, a FLoc-morphism ψ between spacetimes with frames e and e' obeys $\psi^* e'^\mu = e^\mu$ (see Sect. 2.1). The category FLoc provides a minimal setting for general locally covariant theories and was introduced in order to discuss the spin-statistics connection [17, 18]; it has also found use in the perturbative programme [36]. For our purposes the key point is that the restricted Lorentz group $\mathcal{L}_0$ acts on FLoc, by modifying the coframe as $e \mapsto \Lambda e$, where $(\Lambda e)^\mu = \Lambda^\mu{}_\nu e^\nu$. This group action leaves the metric and (time)-orientation unchanged, and physical theories should be covariant with respect to it.

Lorentz covariance in this sense is neither an internal nor a spacetime symmetry (indeed, it maps between backgrounds that are not generally linked by any morphism of FLoc). A similar situation occurs for rigid scaling, which also acts on FLoc and the category Loc often used in locally covariant QFT; not all theories display rigid scale covariance, but it is useful to be able to distinguish and analyze those that do. We therefore make a systematic analysis of theories that are covariant under a group action on BkGrnd (Sect. 2) and illustrate it using rigid scaling (Sect. 3) before passing to the discussion of Lorentz symmetry and our main result (Sect. 4).

The outline is as follows. Suppose a group G acts functorially on the category BkGrnd so that $g \in G$ maps any spacetime $\boldsymbol{M}$ to some ${}^g\boldsymbol{M}$ and each morphism $\psi : \boldsymbol{M} \to \boldsymbol{N}$ to some ${}^g\psi : {}^g\boldsymbol{M} \to {}^g\boldsymbol{N}$, with the identity acting trivially and ${}^{gh}\boldsymbol{M} = {}^g({}^h\boldsymbol{M})$, ${}^{gh}\psi = {}^g({}^h\psi)$. Given a theory $\mathcal{A} : \mathsf{BkGrnd} \to \mathsf{Phys}$, each element $g \in G$ determines a new theory ${}^g\mathcal{A}$ obtained by defining ${}^g\mathcal{A}(\boldsymbol{M}) = \mathcal{A}({}^g\boldsymbol{M})$ and ${}^g\mathcal{A}(\psi) = \mathcal{A}({}^g\psi)$.[1] We say that $\mathcal{A}$ is G-covariant if all these theories are physically equivalent, meaning that there is a natural isomorphism between $\mathcal{A}$ and each ${}^g\mathcal{A}$. As will be shown, these isomorphisms determine a group 2-cocycle of G with coefficients in the (potentially nonabelian) gauge group $\mathrm{Aut}(\mathcal{A})$. It turns out (Theorem 6) that this 2-cocycle is intrinsic to $\mathcal{A}$; any other system of isomorphisms between $\mathcal{A}$ and the ${}^g\mathcal{A}$ results in a cohomologous 2-cocycle; in other words the G-covariance determines a distinguished cohomology class $[\mathcal{A}]_G \in H^2(G, \mathrm{Aut}(\mathcal{A}))$. Associated with this class is a canonical group extension E of G by $\mathrm{Aut}(\mathcal{A})$, under which the fields of the theory transform in multiplets (Theorem 8). A key question is whether such E-multiplets might contain inequivalent submultiplets for the action of G that are mixed under the action of E. This can be excluded (for irreducible G-multiplets) if E is simply a direct product $E = \mathrm{Aut}(\mathcal{A}) \times G$, which holds if $[\mathcal{A}]_G$ is trivial (Corollary 9).

Theorem 11 uses this general analysis to prove that any theory defined on FLoc obeying the timeslice property, additivity and dynamical local Lorentz invariance is $\mathcal{S}$-covariant with a trivial 2-cocycle, where $\mathcal{S}$ is the universal covering of the restricted Lorentz group. These conditions will be described in detail later; the first two are standard and express the existence of dynamics and the ability to build up the theory from sub-spacetimes (as expected for a theory of quantum fields). The third uses relative Cauchy evolution [4], the dynamical response to perturbations in the background structures, to

[1] This action is written contravariantly, ${}^{gh}\mathcal{A} = {}^h({}^g\mathcal{A})$, to avoid a proliferation of inverses.

express invariance with respect to local changes of frame. Theorem 11 is proved by an explicit geometrical construction, using smooth deformations of the background frame to connect a given framed spacetime $\boldsymbol{M}$ to ${}^{\Lambda}\boldsymbol{M}$, which differs from $\boldsymbol{M}$ only by a rigid Lorentz frame rotation. The timeslice property induces an isomorphism between $\mathcal{A}(\boldsymbol{M})$ and $\mathcal{A}({}^{\Lambda}\boldsymbol{M})$ which depends on the deformation only via its homotopy class (as a result of local dynamical Lorentz invariance) so the covering group $\mathcal{S}$ enters in a manner reminiscent of Dirac's belt trick. One then shows these individual isomorphisms implement $\mathcal{S}$-covariance with trivial 2-cocycle. As a consequence, the extended group is a direct product $E = \mathrm{Aut}(\mathcal{A}) \times \mathcal{S}$, and all fields of the theory transform in multiplets under true representations of $\mathcal{S}$. Further, the possibility of noninteger spin can be related to the structure of the centre of the global gauge group. Thus, a theory of observables alone, with trivial global gauge group, can only admit integer spin; the same is true, for different reasons, of any theory initially defined on the category Loc of globally hyperbolic spacetimes.

We have mentioned that Theorem 11 drops some crucial assumptions of the CM theorem. For example, the CM theorem requires interaction because some free Minkowski theories have symmetries that mix fields of different spin. Theorem 11 replaces this by the assumption that the theory can be formulated in all spacetimes in a locally covariant fashion and that the symmetries under discussion are present in general spacetimes. To illustrate the point, consider free scalar and Proca fields ϕ and A with equal nonzero mass in $n = 4$ Minkowski space. The current $j_{ab} = \phi \overleftrightarrow{\partial}_a A_b$ is conserved on-shell and generates a group action that mixes ϕ and A in a nonlocal fashion [29, Sect. 5]. However, this symmetry does not extend to curved spacetimes[2] and so there is no contradiction with Theorem 11: from a curved spacetime perspective, this higher spin symmetry is a quirk of the vacuum representation of the Minkowski theory. Similar remarks apply to factorizing models in $n = 2$ Minkowski space that evade the CM theorem [33]. Further comments and extensions are discussed in Sect. 5.

2. G-Covariance

2.1. Motivating examples. Three categories of spacetimes will be needed: Loc, FLoc and $\mathsf{SpinLoc}$. Loc is the category of oriented globally hyperbolic spacetimes [4] with objects $\boldsymbol{M} = (\mathcal{M}, g, \mathfrak{o}, \mathfrak{t})$ comprising a smooth paracompact manifold $\mathcal{M}$ of fixed dimension $n \geq 2$ and at most finitely many components, a smooth Lorentzian metric g on $\mathcal{M}$ with signature $+ - \cdots -$, and an orientation $\mathfrak{o}$ and time-orientation $\mathfrak{t}$ represented as equivalence classes of nonvanishing n-forms or time-like 1-forms. It is required that $\boldsymbol{M}$ be globally hyperbolic: every $J^+_{\boldsymbol{M}}(p) \cap J^-_{\boldsymbol{M}}(q)$ is compact ($p, q \in \boldsymbol{M}$) and there are no closed timelike curves; equivalently $\boldsymbol{M}$ has Cauchy surfaces. Morphisms in Loc are smooth isometric embeddings, preserving the orientation and time-orientation, and with causally convex image; thus all causal relations between points in the image of a morphism are already present in its domain.

FLoc is the category of framed globally hyperbolic spacetimes,[3] the objects of which are all pairs $\mathfrak{M} = (\mathcal{M}, e)$, where $\mathcal{M}$ is a smooth n-dimensional manifold with smooth global coframe $e = (e^\nu)_{\nu=0}^{n-1}$ such that

$$\mathfrak{F}_{\mathsf{L}}(\mathcal{M}, e) := (\mathcal{M}, \eta_{\mu\nu} e^\mu \otimes e^\nu, [e^0 \wedge \cdots \wedge e^{n-1}], [e^0])$$

[2] Replacing ∂_a by covariant derivatives, $\nabla^a j_{ab} = -\phi R_{bc} A^c$ on-shell, for example; in general there is no conserved rank-2 combination of ϕ and A and their derivatives.

[3] See [17,18]; a related category appears in [12, Ch. 6].

defines an object of Loc. Here $\eta_{\mu\nu}e^\mu \otimes e^\nu$ is the *e-metric*, where $\eta = \mathrm{diag}(+1, -1, \ldots, -1)$, $[e^0]$ is the equivalence class of nonvanishing e-timelike covector fields containing e^0, and $[e^0 \wedge \cdots \wedge e^{n-1}]$ is the equivalence class of nonvanishing n-forms containing $e^0 \wedge \cdots \wedge e^{n-1}$. Thus we form the spacetime metric and (time-)orientation from the coframe and require the resulting structure to be globally hyperbolic. A morphism $\psi : (\mathcal{M}, e) \to (\mathcal{M}', e')$ in FLoc is determined by a smooth map $\psi : \mathcal{M} \to \mathcal{M}'$ that induces a Loc-morphism $\mathcal{F}_\mathsf{L}(\mathcal{M}, e) \to \mathcal{F}_\mathsf{L}(\mathcal{M}', e')$ and obeys $\psi^*e' = e$. In this way, $\mathcal{F}_\mathsf{L}$ is promoted to a functor $\mathcal{F}_\mathsf{L} : \mathsf{FLoc} \to \mathsf{Loc}$.

Finally, $\mathsf{SpinLoc}$ is the category of globally hyperbolic spacetimes with spin structure, restricting to those for which the spin bundle is trivial (which includes all orientable globally hyperbolic spacetimes in $n = 4$ dimensions [26]). Let $\mathcal{S}$ be the universal cover of the restricted Lorentz group $\mathcal{L}_0 = \mathrm{SO}_0(1, n-1)$, with covering homomorphism $\pi : \mathcal{S} \to \mathcal{L}_0$. In brief,[4] the objects of $\mathsf{SpinLoc}$ are exactly those of FLoc, but a $\mathsf{SpinLoc}$ morphism from $(\mathcal{M}, e)$ to $(\mathcal{M}', e')$ is a pair (ψ, Ξ) where the Loc-morphism $\psi : \mathcal{F}_\mathsf{L}(\mathcal{M}, e) \to \mathcal{F}_\mathsf{L}(\mathcal{M}', e')$ and map $\Xi \in C^\infty(\mathcal{M}, \mathcal{S})$ obey $\psi^*e' = \pi(\Xi)e$. Composition of morphisms is given by $(\psi', \Xi') \circ (\psi, \Xi) = (\psi' \circ \psi, (\psi^*\Xi')\Xi)$, where $((\psi^*\Xi')\Xi)(p) = \Xi'(\psi(p))\Xi(p)$.

There is a functor $\mathcal{F}_\mathsf{S} : \mathsf{FLoc} \to \mathsf{SpinLoc}$ given by $\mathcal{F}_\mathsf{S}(\boldsymbol{\mathcal{M}}) = \boldsymbol{\mathcal{M}}$, $\mathcal{F}_\mathsf{S}(\psi) = (\psi, 1)$, and a functor $\mathcal{U} : \mathsf{SpinLoc} \to \mathsf{Loc}$ given by $\mathcal{U}(\boldsymbol{\mathcal{M}}) = \mathcal{F}_\mathsf{L}(\boldsymbol{\mathcal{M}})$, $\mathcal{U}(\psi, \Xi) = \psi$, with composition $\mathcal{U} \circ \mathcal{F}_\mathsf{S} = \mathcal{F}_\mathsf{L}$. Therefore any theory $\mathcal{A}$ on Loc induces theories $\mathcal{A} \circ \mathcal{U}$ on $\mathsf{SpinLoc}$ and $\mathcal{A} \circ \mathcal{F}_\mathsf{L}$ on FLoc, while any theory $\mathcal{B}$ on $\mathsf{SpinLoc}$ (e.g., the Dirac field [39]) induces a theory $\mathcal{B} \circ \mathcal{F}_\mathsf{S}$ on FLoc.

The category FLoc has a number of advantages: it is an operationally motivated arena for curved spacetime physics in which measurements are made with respect to a system of rods and clocks. Unlike Loc, it admits theories of both integer and noninteger spin; unlike $\mathsf{SpinLoc}$, the objects and morphisms are given entirely in terms of observable structures.

All three categories admit physically relevant group actions:

Example 1. The multiplicative group $\mathbb{R}^+$ acts on Loc by rigid metric scaling: for each $\lambda \in \mathbb{R}^+$, there is a functor $\mathcal{R}(\lambda) : \mathsf{Loc} \to \mathsf{Loc}$ defined on objects by

$$\mathcal{R}(\lambda)(\mathcal{M}, g, \mathfrak{o}, \mathfrak{t}) = (\mathcal{M}, \lambda^2 g, \mathfrak{o}, \mathfrak{t})$$

and so that $\mathcal{R}(\lambda)(\psi)$ has the same underlying map of manifolds as ψ for any morphism ψ of Loc. The length of a curve in $\mathcal{R}(\lambda)(\boldsymbol{M})$ is λ times its length in $\boldsymbol{M}$; alternatively, one may think of $\mathcal{R}(\lambda)(\boldsymbol{M})$ as a version of $\boldsymbol{M}$ in which the fundamental unit of length has been divided by λ. Given a theory $\mathcal{A} : \mathsf{Loc} \to \mathsf{Phys}$ we obtain a new theory $\mathcal{A} \circ \mathcal{R}(\lambda)$ for each $\lambda \in \mathbb{R}^+$; the theory is (rigidly) scale covariant if all these theories are equivalent, i.e., naturally isomorphic functors—see Sect. 3 for a specific example. Of course scaling acts in similar ways on both FLoc and $\mathsf{SpinLoc}$.

Example 2. The Lorentz group $\mathcal{L}$ acts functorially on FLoc by $\mathcal{T}(\Lambda)(\mathcal{M}, e) = (\mathcal{M}, \Lambda e)$, where $(\Lambda e)^\mu = \Lambda^\mu{}_\nu e^\nu$ is the Lorentz-transformed coframe; the action of $\mathcal{T}(\Lambda)$ on morphisms is defined so as to preserve the underlying map of manifolds. It is clear that $\mathcal{T}(\Lambda'\Lambda) = \mathcal{T}(\Lambda') \circ \mathcal{T}(\Lambda)$. In the present paper we only consider the action of the restricted Lorentz group $\mathcal{L}_0$ (i.e., the identity component of $\mathcal{L}$) for which $\mathcal{F}_\mathsf{L}(\mathcal{T}(\Lambda))$ is the identity; the discrete transformations will be discussed elsewhere. A theory $\mathcal{A} : \mathsf{FLoc} \to \mathsf{Phys}$ is (rigidly) Lorentz covariant if $\mathcal{A}$ and ${}^\Lambda\mathcal{A} := \mathcal{A} \circ \mathcal{T}(\Lambda)$ are equivalent for all $\Lambda \in \mathcal{L}_0$.

[4] The presentation here is streamlined and will be described in detail elsewhere [13].

Example 3. The universal cover $\mathcal{S}$ of $\mathcal{L}_0$ acts on FLoc by means of $\mathcal{T} \circ \pi$. It also acts on $\mathsf{SpinLoc}$, by means of functors $\mathcal{S}(S)$ agreeing with $(\mathcal{T} \circ \pi)(S)$ on objects and giving $\mathcal{S}(S)(\psi, \Xi) = (\psi, S\Xi S^{-1})$ on morphisms. All theories $\mathcal{A} : \mathsf{SpinLoc} \to \mathsf{Phys}$ are $\mathcal{S}$-covariant via $\mathbb{S}$: to each $S \in \mathcal{S}$ there is a natural isomorphism $\eta(S) : \mathcal{A} \stackrel{\cdot}{\to} {}^S\mathcal{A}$ with components $\eta(S)_{\mathcal{M}} = \mathcal{A}(\mathrm{id}_{\mathcal{F}_{\mathsf{L}}(\mathcal{M})}, S)$, as shown by the calculation

$$\eta(S)_{\mathcal{M}'}\mathcal{A}(\psi, \Xi) = \mathcal{A}(\psi, S\Xi) = \mathcal{A}(\psi, S\Xi S^{-1})\mathcal{A}(\mathrm{id}_{\mathcal{F}_{\mathsf{L}}(\mathcal{M})}, S) = {}^S\mathcal{A}(\psi, \Xi)\eta_{\mathcal{M}}(S)$$

for any $(\psi, \Xi) : \mathcal{M} \to \mathcal{M}'$. The corresponding 2-cocycle is trivial (see below).

2.2. General analysis. The examples above motivate the study of the following situation. Let G be any group and suppose there is a homomorphism $\mathcal{T} : G \to \mathrm{Aut}(\mathsf{C})$, where C is a category and $\mathrm{Aut}(\mathsf{C})$ is the group of invertible functors from C to itself. Clearly $\mathcal{T}(g)$ has inverse $\mathcal{T}(g^{-1})$ and so every morphism of C is contained in the image of each $\mathcal{T}(g)$. For brevity, we will often write the action of $\mathcal{T}(g)$ on objects C and morphisms γ of C by ${}^gC := \mathcal{T}(g)(C)$, and ${}^g\gamma := \mathcal{T}(g)(\gamma)$.

Definition 4. A functor $\mathcal{A} : \mathsf{C} \to \mathsf{C}'$ is *G-covariant*[5] *(via $\mathcal{T}$)* if all the functors ${}^g\mathcal{A} = \mathcal{A} \circ \mathcal{T}(g)$ are naturally isomorphic; any family $\eta(g) : \mathcal{A} \stackrel{\cdot}{\to} {}^g\mathcal{A}$ $(g \in G)$ of natural isomorphisms with $\eta(1) = \mathrm{id}_{\mathcal{A}}$ is an *implementation* of the G-covariance.

Here C' is any category. It will be shown that all implementations of a G-covariance are equivalent in the sense of nonabelian cohomology, and correspond to a uniquely determined element of the second cohomology set $H^2(G, \mathrm{Aut}(\mathcal{A}))$.

Let us briefly recall that if G and A are (not necessarily abelian) groups then a 2-cochain of G with coefficients in A is a pair (ξ, ϕ) consisting of maps $\xi : G \times G \to A$ and $\phi : G \to \mathrm{Aut}(A)$; (ξ, ϕ) is a 2-cocycle if

$$\phi(g')\phi(g)\phi(g'g)^{-1} = \mathrm{ad}(\xi(g', g)) \qquad (g', g \in G), \tag{1}$$

$$\xi(g'', g')\xi(g''g', g) = \phi(g'')(\xi(g', g))\xi(g'', g'g) \qquad (g'', g', g \in G) \tag{2}$$

and the set of such 2-cocycles is denoted $Z^2(G, A)$. Two 2-cocycles $(\xi, \phi), (\widetilde{\xi}, \widetilde{\phi}) \in Z^2(G, A)$ are cohomologous precisely if there is a map $\zeta : G \to A$ such that

$$\widetilde{\phi}(g) = \mathrm{ad}(\zeta(g)) \circ \phi(g) \quad \text{and} \quad \widetilde{\xi}(g', g) = \zeta(g')\phi(g')(\zeta(g))\xi(g', g)\zeta(g'g)^{-1} \tag{3}$$

for all $g', g \in G$. The corresponding equivalence classes form the cohomology set $H^2(G, A)$, with the class of the trivial 2-cocycle $(1_A, \mathrm{id}_A)$ as a distinguished element making $H^2(G, A)$ a pointed set. Here $1_A(g', g) = 1 \in A$ for all $g', g \in G$. Cocycles of the form $(1_A, \phi)$, where ϕ is (necessarily) a homomorphism are called *neutral*, as are the corresponding cohomology classes. A 2-cocycle (ξ, ϕ) is *normalized* if $\phi(1) = 1$ and $\xi(g, 1) = \xi(1, g) = 1$ for all $g \in G$.

With these definitions established, our first result is:

Theorem 5. *Any implementation η of a G-covariance of $\mathcal{A} : \mathsf{C} \to \mathsf{C}'$ determines a normalized* 2*-cocycle* $(\xi, \phi) \in Z^2(G, \mathrm{Aut}(\mathcal{A}))$ *given by*

$$\xi(g', g)_{g'gC} = \eta(g')_{{}^gC}\eta(g)_C\eta(g'g)_C^{-1} \qquad (g', g \in G, C \in \mathsf{C}) \tag{4}$$

$$\phi(g)(\alpha)_{{}^gC} = \eta(g)_C\alpha_C\eta(g)_C^{-1} \qquad (\alpha \in \mathrm{Aut}(\mathcal{A}), g \in G, C \in \mathsf{C}). \tag{5}$$

[5] There is an unhappy collision of terminology: $\mathcal{A}$ is a covariant functor in the usual category theory sense; G-covariance is an additional and somewhat different property.

Proof. Eqs. (4) and (5) are easily seen to define automorphisms $\xi(g', g)_C$ and $\phi(g)(\alpha)_C$ of $\mathcal{A}(C)$ for every $C \in \mathsf{C}$ by the properties of $\mathcal{T}(g)$ described above. The rest of the proof is broken into several calculations.
Naturality and automorphism properties of ϕ: Suppose that $\gamma : C \to C'$. Then

$$\begin{aligned}\mathcal{A}({}^g\gamma)\phi(g)(\alpha)_{{}^gC} &= \eta(g)_{C'}\mathcal{A}(\gamma)\alpha_C\eta(g)_C^{-1} = \eta(g)_{C'}\alpha_{C'}\mathcal{A}(\gamma)\eta(g)_C^{-1}\\ &= \eta(g)_{C'}\alpha_{C'}\eta(g)_{C'}^{-1}\mathcal{A}({}^g\gamma) = \phi(g)(\alpha)_{{}^gC'}\mathcal{A}({}^g\gamma)\end{aligned}$$

so each $\phi(g)(\alpha) \in \mathrm{Aut}(\mathcal{A})$. It is clear from (5) that $\phi(g)(\alpha\beta) = \phi(g)(\alpha)\phi(g)(\beta)$ so $\phi : g \to \phi(g)$ is a map from G to $\mathrm{Aut}(\mathrm{Aut}(\mathcal{A}))$.
Naturality of $\xi(g', g)$: This is proved by calculating, for arbitrary $\gamma : C \to C'$,

$$\begin{aligned}\xi(g', g)_{g'gC'}\mathcal{A}({}^{g'g}\gamma) &= \eta(g')_{{}^gC'}\eta(g)_{C'}\eta(g'g)_{C'}^{-1}\mathcal{A}({}^{g'g}\gamma)\\ &= \eta(g')_{{}^gC'}\eta(g)_{C'}\mathcal{A}(\gamma)\eta(g'g)_C^{-1}\\ &= \eta(g')_{{}^gC'}\mathcal{A}({}^g\gamma)\eta(g)_C\eta(g'g)_C^{-1}\\ &= \mathcal{A}({}^{g'g}\gamma)\eta(g')_{{}^gC}\eta(g)_C\eta(g'g)_C^{-1}\\ &= \mathcal{A}({}^{g'g}\gamma)\xi(g', g)_{g'gC}.\end{aligned}$$

Cocycle property: Normalization of ϕ is obvious from (5); $\xi(g, 1) = \xi(1, g) = 1$ is immediate using $\eta(1) = \mathrm{id}_{\mathcal{A}}$. Let $\alpha \in \mathrm{Aut}(\mathcal{A})$ and compute

$$\begin{aligned}\mathrm{ad}(\xi(g', g))(\alpha)_{g'gC} &= \xi(g', g)_{g'gC}\alpha_{g'gC}\xi(g', g)_{g'gC}^{-1}\\ &= \eta(g')_{{}^gC}\eta(g)_C\eta(g'g)_C^{-1}\alpha_{g'gC}\eta(g'g)_C\eta(g)_C^{-1}\eta(g')_{{}^gC}^{-1}\\ &= \phi(g')(\phi(g)(\phi(g'g)^{-1}(\alpha)))_{g'gC}\end{aligned}$$

for any $g', g \in G$ and $C \in \mathsf{C}$, so $\mathrm{ad}\,\xi(g', g) = \phi(g')\phi(g)\phi(g'g)^{-1}$ as required by (1). Finally, let $g'', g', g \in G$ and $C \in \mathsf{C}$ be arbitrary, then

$$\begin{aligned}(\xi(g'', g')\xi(g''g', g))_{g''g'gC} &= \eta(g'')_{g'gC}\eta(g')_{{}^gC}\eta(g)_C\eta(g''g'g)_C^{-1}\\ &= \eta(g'')_{g'gC}\xi(g', g)_{g'gC}\eta(g'')_{g'gC}^{-1}\xi(g'', g'g)_{g''g'gC}\\ &= \phi(g'')(\xi(g', g))_{g''g'gC}\xi(g'', g'g)_{g''g'gC}\end{aligned}$$

so the cocycle condition (2) also holds. Thus $(\xi, \phi) \in Z^2(G, A)$. $\square$

For example, the 2-cocycle mentioned in Example 3 is trivial, because $\eta(S'S)_{\mathcal{M}} = \mathcal{A}(\mathrm{id}_{\mathcal{F}_{\mathsf{L}}(S\mathcal{M})}, S')\mathcal{A}(\mathrm{id}_{\mathcal{F}_{\mathsf{L}}(\mathcal{M})}, S) = \eta(S')_{S\mathcal{M}}\eta(S)_{\mathcal{M}}$, and $\eta(S)_{\mathcal{M}}\alpha_{\mathcal{M}} = \alpha_{S\mathcal{M}}\eta(S)_{\mathcal{M}}$ by naturality of $\alpha \in \mathrm{Aut}(\mathcal{A})$ and the definition of $\eta(S)$.

The 2-cocycle given by Theorem 5 is intrinsic to $\mathcal{A}$.

Theorem 6. *If $\mathcal{A}$ is G-covariant, the 2-cocycles of its implementations form a distinguished cohomology class $[\mathcal{A}]_G \in H^2(G, \mathrm{Aut}(\mathcal{A}))$.*

Proof. We show that all implementations induce cohomologous 2-cocycles, and all elements of the corresponding cohomology class arise from implementations.

First, let $g :\mapsto \eta(g)$ be an implementation, let $\zeta : G \to \mathrm{Aut}(\mathcal{A})$ be any map and set $\widetilde{\eta}(g)_C = \zeta(g)_{{}^gC}\eta(g)_C$. Then $g \mapsto \widetilde{\eta}(g)$ also implements the G-covariance, and η and

$\widetilde{\eta}$ define cohomologous 2-cocycles. To see this, note that each $\widetilde{\eta}(g)_C : \mathcal{A}(C) \to \mathcal{A}({}^gC)$ is certainly an isomorphism. If $\gamma : C \to C'$ then

$$\widetilde{\eta}(g)_{C'}\mathcal{A}(\gamma) = \zeta(g)_{{}^gC'}\eta(g)_{C'}\mathcal{A}(\gamma) = \zeta(g)_{{}^gC'}\mathcal{A}({}^g\gamma)\eta(g)_C = \mathcal{A}({}^g\gamma)\zeta(g)_{{}^gC}\eta(g)_C \\ = \mathcal{A}({}^g\gamma)\widetilde{\eta}(g)_C,$$

which establishes naturality, so $g \mapsto \widetilde{\eta}(g)$ implements the G-covariance. The corresponding 2-cocycle $(\widetilde{\xi}, \widetilde{\phi})$ is computed as follows:

$$\widetilde{\phi}(g)(\alpha)_{{}^gC} = \zeta(g)_{{}^gC}\eta(g)_C\alpha_C\eta(g)_C^{-1}\zeta(g)_{{}^gC}^{-1} = (\mathrm{ad}\,\zeta(g))(\phi(g)(\alpha))_{{}^gC},$$

while

$$\begin{aligned}\widetilde{\xi}(g', g)_{g'{}^gC} &= \zeta(g')_{g'{}^gC}\eta(g')_{{}^gC}\zeta(g)_{{}^gC}\eta(g)_C\eta(g'g)_C^{-1}\zeta(g'g)_{g'{}^gC}^{-1} \\ &= \zeta(g')_{g'{}^gC}\phi(g')(\zeta(g))_{g'{}^gC}\eta(g')_{{}^gC}\eta(g)_C\eta(g'g)_C^{-1}\zeta(g'g)_{g'{}^gC}^{-1} \\ &= (\zeta(g')\phi(g')(\zeta(g))\xi(g', g)\zeta(g'g)^{-1})_{g'{}^gC}.\end{aligned}$$

The conditions in (3) are met so the 2-cocycles are cohomologous.

To prove the result, we suppose that implementations η and $\widetilde{\eta}$ have been given. If the morphisms $\zeta(g)_{{}^gC} := \widetilde{\eta}(g)_C\eta(g)_C^{-1}$ form the components of an automorphism $\zeta(g) \in \mathrm{Aut}(\mathcal{A})$ for each g, then the first part of the proof demonstrates that the implementations induce the same cohomology class. As the maps $\zeta(g)_{{}^gC}$ are clearly isomorphisms it remains to check naturality: if $\gamma : C \to C'$, then

$$\zeta(g)_{{}^gC'}\mathcal{A}({}^g\gamma) = \widetilde{\eta}(g)_{C'}\eta(g)_{C'}^{-1}\mathcal{A}({}^g\gamma) = \widetilde{\eta}(g)_{C'}\mathcal{A}(\gamma)\eta(g)_C^{-1} = \mathcal{A}({}^g\gamma)\widetilde{\eta}(g)_C\eta(g)_C^{-1} \\ = \mathcal{A}({}^g\gamma)\zeta(g)_{{}^gC},$$

which establishes naturality as every morphism is the image of $\mathcal{T}(g)$. Finally, if $(\widetilde{\xi}, \widetilde{\phi}) \sim (\xi, \phi)$ then one has $\zeta : G \to \mathrm{Aut}(\mathcal{A})$ obeying (3), whereupon $\widetilde{\eta}(g)$, defined using ζ as above, implements the G-covariance with cocycle $(\widetilde{\xi}, \widetilde{\phi})$. $\square$

If $[\mathcal{A}]_G \in H^2(G, \mathrm{Aut}(\mathcal{A}))$ is trivial, then one may choose an implementation corresponding to the trivial cocycle $(1_A, \mathrm{id}_A)$. In this case, one has

$$\eta(g)_C\alpha_C = \alpha_{{}^gC}\eta(g)_C, \quad \eta(g'g)_C = \eta(g')_{{}^gC}\eta(g)_C, \qquad (g', g \in G, C \in \mathsf{C}). \tag{6}$$

Returning to the general case, suppose $\mathcal{A} : \mathsf{C} \to \mathsf{C}'$ is G-covariant and choose an implementation $g \mapsto \eta(g)$ with normalized 2-cocycle $(\xi, \phi) \in Z^2(G, \mathrm{Aut}(\mathcal{A}))$. The 2-cocycle induces a group extension of G by $\mathrm{Aut}(\mathcal{A})$, described by a short exact sequence of group homomorphisms

$$1 \to \mathrm{Aut}(\mathcal{A}) \to E \xrightarrow{q} G \to 1, \tag{7}$$

where the extension $E = \mathrm{Aut}(\mathcal{A}) \times G$ as a set, and is equipped with the product

$$(a', g')(a, g) = (a'\phi(g')(a)\xi(g', g), g'g) \tag{8}$$

for which $(1, 1)$ is the identity. The unlabelled map $\mathrm{Aut}(\mathcal{A}) \to E$ in (7) is $a \mapsto (a, 1)$, and embeds $\mathrm{Aut}(\mathcal{A})$ as a normal subgroup of E, while $q(a, g) = g$ and realizes G as the quotient $G \cong E/\mathrm{Aut}(\mathcal{A})$. See, e.g., [1,11]. The group extension is determined by the cohomology class $[\mathcal{A}]_G$ up to a suitable equivalence of extensions. Some familiar cases arise as follows: the trivial cocycle gives the direct product $\mathrm{Aut}(\mathcal{A}) \times G$; a neutral cocycle $(1, \phi)$ gives the semidirect product $\mathrm{Aut}(\mathcal{A}) \rtimes_\phi G$; if $\mathrm{Aut}(\mathcal{A})$ is abelian then $(\xi, 1)$ gives a central extension.

A G-covariant theory is also covariant under the corresponding group extension, which almost trivialises the cocycle (one might say that it is neutralised).

Theorem 7. *If $\mathcal{A} : \mathsf{C} \to \mathsf{C}'$ is G-covariant via $\mathcal{T}$, then $\mathcal{A}$ is E-covariant via $\mathcal{T} \circ q$, with a neutral cocycle in $Z^2(E, \mathrm{Aut}(\mathcal{A}))$.*

Proof. (Sketch) The E-covariance is implemented by $(\alpha, g) \mapsto \rho(\alpha, g)$, where $\rho(\alpha, g)_C = \alpha_{\mathcal{T}(g)(C)}\eta(g)_C$. The 2-cocycle is $(1, \varphi)$, with $\varphi(\alpha, g) = \mathrm{ad}\,\alpha \circ \phi(g)$. □

2.3. Multiplets of locally covariant fields for G-covariant theories. Consider a locally covariant QFT given as a functor $\mathcal{A} : \mathsf{BkGrnd} \to \mathsf{Alg}$, where BkGrnd is $(\mathsf{Spin})\mathsf{Loc}$ or FLoc, and Alg is the category of unital $*$-algebras and unit-preserving $*$-monomorphisms. Let $\mathcal{D} : \mathsf{BkGrnd} \to \mathsf{Set}$ be the functor assigning to each $C \in \mathsf{BkGrnd}$ the set of smooth complex-valued compactly supported test functions on the underlying manifold of C, and to each morphism ψ, the corresponding push-forward $\mathcal{D}(\psi) = \psi_*$. Let $\mathcal{V}$ be the forgetful functor $\mathcal{V} : \mathsf{Alg} \to \mathsf{Set}$. By definition, a *locally covariant quantum field* [4,24,41] is a natural transformation $\Phi : \mathcal{D} \stackrel{.}{\to} \mathcal{V} \circ \mathcal{A}$ and the set of all such fields $\mathrm{Fld}(\mathcal{A})$ forms a unital $*$-algebra in a natural way [14]: e.g., $(\mu\Phi + \nu\Psi)_C(f) = \mu\Phi_C(f) + \nu\Psi_C(f)$ and $(\Phi\Psi)_C(f) = \Phi_C(f)\Psi_C(f)$ ($\mu, \nu \in \mathbb{C}$, $f \in \mathcal{D}(C)$) define the linear combination and product of $\Phi, \Psi \in \mathrm{Fld}(\mathcal{A})$. The unit field is $\mathbf{1}_C(f) = \mathbf{1}_{\mathcal{A}(\boldsymbol{M})}$ for all $f \in \mathcal{D}(C)$.[6]

An advantage of theories defined on FLoc is that one need only consider single-component fields in $\mathrm{Fld}(\mathcal{A})$, whereas on $(\mathsf{Spin})\mathsf{Loc}$ one requires a different functor $\mathcal{D}$ for each tensorial field type. For example, a Proca field theory on Loc describes a field smeared against test one-forms, $A_{\boldsymbol{M}}(\omega)$, whereas the same theory pulled back to FLoc has available four single-component fields A^μ, given by $A^\mu_{(\mathcal{M},e)}(f) := A_{\mathcal{F}_\mathsf{L}(\mathcal{M},e)}(fe^\mu)$. The same can be done for spinor fields on spacetimes in $\mathsf{SpinLoc}$, because the spin bundle is trivial. Fully worked out examples will be given elsewhere [13]. Of course, it is then necessary to discern some structure on the fields, which provides a useful application of G-covariance.

Now suppose that a group G acts functorially on BkGrnd, and that both $\mathcal{A}$ and $\mathcal{D}$ are G-covariant. For simplicity we assume that G-covariance of $\mathcal{D}$ is implemented by a family $\zeta(g)$ with trivial cocycle in $Z^2(G, \mathrm{Aut}(\mathcal{D}))$, i.e.,

$$\zeta(g'g)_C = \zeta(g')_{gC}\zeta(g)_C, \quad \text{and} \quad \zeta(g)_C\alpha_C = \alpha_{gC}\zeta(g)_C$$

for all $g', g \in G$, $\alpha \in \mathrm{Aut}(\mathcal{D})$ and $C \in \mathsf{BkGrnd}$. In this situation, the fields in $\mathrm{Fld}(\mathcal{A})$ transform under both G and $\mathrm{Aut}(\mathcal{A})$.

Theorem 8. *Suppose $\mathcal{A} : \mathsf{BkGrnd} \to \mathsf{Alg}$ and $\mathcal{D} : \mathsf{BkGrnd} \to \mathsf{Set}$ are G-covariant and that the G-covariance of $\mathcal{A}$ is implemented by η, with 2-cocycle (ξ, ϕ), while that of $\mathcal{D}$ is implemented by ζ, with trivial cocycle. Let $\Phi \in \mathrm{Fld}(\mathcal{A})$. Then for each $\alpha \in \mathrm{Aut}(\mathcal{A})$ there is a transformed field $\alpha \cdot \Phi \in \mathrm{Fld}(\mathcal{A})$ defined by*

$$(\alpha \cdot \Phi)_C = \mathcal{V}(\alpha_C)\Phi_C, \qquad (C \in \mathsf{BkGrnd}) \tag{9}$$

[6] Using Set allows for fields depending nonlinearly on the test function. Using the category of vector spaces instead, one obtains a vector space (rather than $*$-algebra) of linear fields.

and for each $g \in G$ there is a transformed field $g * \Phi \in \mathrm{Fld}(\mathcal{A})$ defined by

$$(g * \Phi)_{{}^gC}\zeta(g)_C = \mathcal{V}(\eta(g)_C)\Phi_C, \qquad (C \in \mathsf{BkGrnd}). \tag{10}$$

One has $1_{\mathrm{Aut}\,\mathcal{A}} \cdot \Phi = \Phi = 1_G * \Phi$ for all $\Phi \in \mathrm{Fld}(\mathcal{A})$. The following formulae hold for all $\alpha, \beta \in \mathrm{Aut}(\mathcal{A})$, $g', g \in G$ and $\Phi \in \mathrm{Fld}(\mathcal{A})$:

$$\alpha \cdot (\beta \cdot \Phi) = (\alpha\beta) \cdot \Phi \tag{11}$$
$$g * (\alpha \cdot \Phi) = \phi(g)(\alpha) \cdot (g * \Phi) \tag{12}$$
$$g' * (g * \Phi) = \xi(g', g) \cdot \big((g'g) * \Phi\big). \tag{13}$$

$\mathrm{Fld}(\mathcal{A})$ *carries a true group action ρ of the group extension E of G by $\mathrm{Aut}(\mathcal{A})$ determined by (ξ, ϕ), given by $\rho(\alpha, g)\Phi = \alpha \cdot (g * \Phi)$.*

Proof. The statements concerning the action of $\mathrm{Aut}(\mathcal{A})$ are proved in [15, Sect. 3.2]. Turning to the action of G, we note that (10) defines a transformed field because

$$\begin{aligned}
\mathcal{V}(\mathcal{A}({}^g\gamma))(g * \Phi)_{{}^gC}\zeta(g)_C &= \mathcal{V}(\mathcal{A}({}^g\gamma)\eta(g)_C)\Phi_C = \mathcal{V}(\eta(g)_{C'}\mathcal{A}(\gamma))\Phi_C \\
&= \mathcal{V}(\eta(g)_{C'})\Phi_{C'}\mathcal{D}(\gamma) = (g * \Phi)_{{}^gC'}\zeta(g)_{C'}\mathcal{D}(\gamma) \\
&= (g * \Phi)_{{}^gC'}\mathcal{D}({}^g\gamma)\zeta(g)_C,
\end{aligned}$$

for all $\gamma : C \to C'$ in BkGrnd. As the $\zeta(g)$ are isomorphisms, $g * \Phi \in \mathrm{Fld}(\mathcal{A})$. To prove (12), suppose $g \in G$ and $\alpha \in \mathrm{Aut}(\mathcal{A})$. Calculating

$$\begin{aligned}
\mathcal{V}(\phi(g)(\alpha)_{{}^gC})(g * \Phi)_{{}^gC}\zeta(g)_C &= \mathcal{V}(\eta(g)_C\alpha_C\eta(g)_C^{-1}\eta(g)_C)\Phi_C = \mathcal{V}(\eta(g)_C\alpha_C)\Phi_C \\
&= (g * (\alpha \cdot \Phi))_{{}^gC}\zeta(g)_C,
\end{aligned}$$

we again strip off the isomorphism $\zeta(g)_C$ to obtain the required result. Next,

$$\begin{aligned}
\mathcal{V}(\xi(g', g)_{{}^{g'g}C})((g'g) * \Phi)_{{}^{g'g}C}\zeta(g'g)_C &= \mathcal{V}(\xi(g', g)_{{}^{g'g}C}\eta(g'g)_C)\Phi_C \\
&= \mathcal{V}(\eta(g')_{{}^gC}\eta(g)_C)\Phi_C \\
&= \mathcal{V}(\eta(g')_{{}^gC})(g * \Phi)_{{}^gC}\zeta(g)_C \\
&= (g' * (g * \Phi))_{{}^{g'g}C}\zeta(g')_{{}^gC}\zeta(g)_C \\
&= (g' * (g * \Phi))_{{}^{g'g}C}\zeta(g'g)_C,
\end{aligned}$$

for $g', g \in G$, using the fact that ζ induces a trivial cocycle.

The final statement follows from (12) and (13) by the calculation

$$\begin{aligned}
\rho(\alpha', g')\rho(\alpha, g)\Phi &= \alpha' \cdot \big(g' * (\alpha \cdot (g * \Phi))\big) = \alpha' \cdot \phi(g')(\alpha) \cdot \big(g' * (g * \Phi)\big) \\
&= \big(\alpha'\phi(g')(\alpha)\xi(g', g)\big) \cdot \big((g'g) * \Phi\big) = \rho((\alpha', g')(\alpha, g))\Phi.
\end{aligned}$$

□

For example, the component fields of a Proca field transform in a vector representation of $\mathcal{L}_0$, $\Lambda * A^\mu = (\Lambda^{-1})^\mu{}_\nu A^\nu$. Thus they can be distinguished from the components of a Dirac spinor or four independent scalars.

In general, Theorem 8 allows one to classify fields by the subrepresentations of ρ in which they transform. A subspace of $\mathrm{Fld}(\mathcal{A})$ (or sometimes, a basis for it) carrying an indecomposable subrepresentation of ρ will be called an *E-multiplet*, augmenting the description with attributes of the subrepresentation (e.g., irreducibility) as appropriate.

The same can be done for the actions of $\mathrm{Aut}(\mathcal{A})$ and G (in the latter case, allowing generalized multiplier representations according to (13)) and referring to $\mathrm{Aut}(\mathcal{A})$- and G-multiplets respectively. One multiplet can be contained in another, if the latter is reducible. Note also that if $\zeta(g)$ commutes with complex conjugation, then the conjugate field $\Phi^\dagger$ to $\Phi \in \mathrm{Fld}(\mathcal{A})$ defined by $\Phi^\dagger_C(f) = \Phi_C(\overline{f})^*$ obeys $g * \Phi^\dagger = (g * \Phi)^\dagger$ and transforms in the complex conjugate representation of that in which Φ transforms. Thus, self-adjoint fields transform in self-conjugate multiplets.

The general structure raises the possibility that distinct G-multiplets can be mixed within a larger E-multiplet. This can be excluded in some circumstances:

Corollary 9. *Under the hypotheses of Theorem 8, suppose additionally that* $[\mathcal{A}]_G \in H^2(G, \mathrm{Aut}(\mathcal{A}))$ *is trivial, so* $E = \mathrm{Aut}(\mathcal{A}) \times G$. *Then no inequivalent irreducible nontrivial G-multiplets can be mixed by the action of E.*

Proof. Let (σ_i, U_i) $(i = 1, 2)$ be irreducible G-representations arising as G-multiplets, i.e., there are linear injections $\iota_i : U_i \to \mathrm{Fld}(\mathcal{A})$ and surjections $\pi_i : \mathrm{Fld}(\mathcal{A}) \to U_i$ so that $\pi_i \rho(1, g) = \sigma_i(g)\pi_i$, $\rho(1, g)\iota_i = \iota_i\sigma_i(g)$, and $\pi_i\iota_i = \mathrm{id}_{U_i}$. If the multiplets mix, there is $e \in E$, which can be taken without loss in the form $e = (\alpha, 1)$, so that $Q = \pi_1\rho(e)\iota_2$ and $R = \pi_2\rho(e)\iota_1$ are not both zero. We assume $R \neq 0$ without loss, and calculate $\sigma_1(g)Q = Q\sigma_2(g)$ and $R\sigma_1(g) = \sigma_2(g)R$, so $\mathrm{Im}\, Q$ and $\ker R$ carry subrepresentations of σ_1, while $\ker Q$ and $\mathrm{Im}\, R$ carry subrepresentations of σ_2. By irreducibility of σ_i, R has trivial kernel and cokernel; hence it is an isomorphism giving $\sigma_1 \simeq \sigma_2$, contradicting the hypothesis. □

Our analysis has been purely algebraic. We comment further on this in Sect. 5; here we mention that, while there are discontinuous finite dimensional representations of many groups including $\mathbb{R}^+$ and $\mathrm{SL}(2, \mathbb{C})$, there are also various ‘automatic continuity’ results. For example, all locally bounded finite-dimensional representations of $\mathrm{SL}(2, \mathbb{C})$ are continuous in the Lie group topology [40].

3. Scaling

As a first illustration we consider the theory of a massless free field with general curvature coupling. The field equation $(\Box + \xi R)\phi = 0$ is invariant under rigid scaling of the metric; we will show that this induces a $\mathbb{R}^+$-covariance via the group action on Loc of Example 1, and that the local Wick powers transform in nontrivial multiplets. We work in $n = 4$ dimensions with $\hbar = c = 1$, so ϕ has dimensions of inverse length. For brevity, we write $\mathcal{R}(\lambda)(\boldsymbol{M}) = \lambda\boldsymbol{M}$.

Construction of the theory The locally covariant description of the QFT is a functor $\mathcal{W} : \mathsf{Loc} \to \mathsf{Alg}$, where each $\mathcal{W}(\boldsymbol{M})$ is the extended algebra of Wick polynomials [24], thereby including the local Wick powers in $\mathrm{Fld}(\mathcal{W})$.

Some preliminaries are required: for each $\boldsymbol{M} \in \mathsf{Loc}$, set $P_{\boldsymbol{M}} = \Box_{\boldsymbol{M}} + \xi R_{\boldsymbol{M}}$ and let $E^{+/-}_{\boldsymbol{M}}$ be the corresponding retarded/advanced Green operators obeying $P_{\boldsymbol{M}} E^{\pm}_{\boldsymbol{M}} f = f$, $\mathrm{supp}\, E^{\pm}_{\boldsymbol{M}} f \subset J^{\pm}_{\boldsymbol{M}}(\mathrm{supp}\, f)$, writing also

$$E_{\boldsymbol{M}}(f, g) = \int_{\boldsymbol{M}} f(p) \left([E^-_{\boldsymbol{M}} - E^+_{\boldsymbol{M}}]g\right)(p) d\mathrm{vol}_{\boldsymbol{M}}(p).$$

Further, choose a $P_{\boldsymbol{M}}$-bisolution $W_{\boldsymbol{M}} \in \mathcal{D}'(\boldsymbol{M} \times \boldsymbol{M})$ obeying

- reality conditions, $\overline{W_{\boldsymbol{M}}(f,g)} = W_{\boldsymbol{M}}(\overline{g},\overline{f})$
- a commutator condition, $W_{\boldsymbol{M}}(f,g) - W_{\boldsymbol{M}}(g,f) = iE_{\boldsymbol{M}}(f,g)$
- a wavefront set constraint, $\mathrm{WF}\,(W_{\boldsymbol{M}}) \subset \mathcal{V}_+(\boldsymbol{M}) \times \mathcal{V}_-(\boldsymbol{M})$,

where $\mathcal{V}_{+/-}(\boldsymbol{M})$ are the closures of the bundles of future/past-pointing causal covectors on $\boldsymbol{M}$.

Given these definitions, the unital $*$-algebra $\mathcal{W}(\boldsymbol{M})$ can be presented in terms of its generators and relations (we will be brief, and refer the reader to e.g. [6,24] for details). There is a unit 1, and the other generators are symbols $:\!\Phi^{\otimes k}\!:_{\boldsymbol{M}}(u)$, labelled by $k \in \mathbb{N}$ and

$$u \in \mathcal{T}^{(k)}(\boldsymbol{M}) := \left\{u \in \mathcal{E}'_{\mathrm{sym}}(\boldsymbol{M}^{\times k}) :\ \mathrm{WF}\,(u) \cap \left(\mathcal{V}_+(\boldsymbol{M})^{\times k} \cup \mathcal{V}_-(\boldsymbol{M})^{\times k}\right) = \emptyset\right\}$$

so that $u \mapsto :\!\Phi^{\otimes k}\!:_{\boldsymbol{M}}(u)$ is linear. Here 'sym' denotes the symmetric subspace and $\mathcal{E}(X)$ is the space of smooth densities on X, while $\mathcal{D}(X)$ are smooth compactly supported functions, so $\mathcal{D}(X)$ is canonically included in $\mathcal{E}'(X)$ without specifying a volume element. The symbols and their adjoints obey relations that are conveniently expressed in terms of a formal power series

$$\mathcal{G}_{\boldsymbol{M}}[f] = \mathbf{1} + \sum_{k=1}^{\infty} \frac{i^k}{k!} :\!\Phi^{\otimes k}\!:_{\boldsymbol{M}}(f^{\otimes k}), \qquad (f \in \mathcal{T}^{(1)}(\boldsymbol{M}))$$

with coefficients in $\mathcal{W}(\boldsymbol{M})$. Writing the $\mathcal{W}(\boldsymbol{M})$-product as $\star_{\boldsymbol{M}}$, the relations are:

- hermiticity, $\mathcal{G}_{\boldsymbol{M}}[f] = \mathcal{G}_{\boldsymbol{M}}[-\overline{f}]^*$
- field equation, $\mathcal{G}_{\boldsymbol{M}}[P_{\boldsymbol{M}} f] = \mathcal{G}_{\boldsymbol{M}}[0]$
- Wick's formula, $\mathcal{G}_{\boldsymbol{M}}[f] \star_{\boldsymbol{M}} \mathcal{G}_{\boldsymbol{M}}[g] = \mathcal{G}_{\boldsymbol{M}}[f+g]e^{-W_{\boldsymbol{M}}(f,g)}$,

understood as identities between formal Taylor coefficients about $f = 0$ (or $f = g = 0$ for Wick's formula) under the rule $:\!\Phi^{\otimes k}\!:_{\boldsymbol{M}}(u) = i^{-k}\langle \delta^k \mathcal{G}_{\boldsymbol{M}}/\delta f^k|_{f=0}, u\rangle$, which are furthermore required to remain valid under linearity and taking limits in $\mathcal{T}^{(\bullet)}(\boldsymbol{M})$ with respect to a suitable topology (or pseudo-topology)—see [9] and [36, Sect. 4.4.2] for a discussion of various possible choices.

This completes the description of the extended algebra $\mathcal{W}(\boldsymbol{M})$ (by contrast, the *unextended* algebra is the unital $*$-subalgebra $\mathcal{A}(\boldsymbol{M})$ generated by $\Phi_{\boldsymbol{M}}(f) := :\!\Phi\!:_{\boldsymbol{M}}(f)$, for $f \in \mathcal{D}(\boldsymbol{M})$). For completeness, however, we spell out the relations in more detail. Hermiticity asserts $:\!\Phi^{\otimes k}\!:_{\boldsymbol{M}}(u) = :\!\Phi^{\otimes k}\!:_{\boldsymbol{M}}(\overline{u})^*$ for all $u \in \mathcal{T}^{(k)}(\boldsymbol{M})$,[7] while Wick's formula corresponds to the relations

$$\begin{aligned}
&:\!\Phi^{\otimes k}\!:_{\boldsymbol{M}}(u) \star_{\boldsymbol{M}} :\!\Phi^{\otimes \ell}\!:_{\boldsymbol{M}}(v) \\
&\qquad = (-i)^{k+\ell}\left\langle \frac{\delta^k}{\delta f^k} \otimes \frac{\delta^\ell}{\delta g^\ell} \mathcal{G}_{\boldsymbol{M}}[f+g]\, e^{-W_{\boldsymbol{M}}(f,g)}\Big|_{f,g=0}, u \otimes v\right\rangle \\
&\qquad = \sum_{j=0}^{\min\{k,\ell\}} :\!\Phi^{\otimes(k+\ell-2j)}\!:_{\boldsymbol{M}}(u \otimes_j v)
\end{aligned}$$

for all $u \in \mathcal{T}^{(k)}(\boldsymbol{M})$, $v \in \mathcal{T}^{(\ell)}(\boldsymbol{M})$. Here, $u \otimes_j v$ is the symmetrized j-times $W_{\boldsymbol{M}}$-contracted tensor product given by $u \otimes_j v = j!\binom{k}{j}\binom{\ell}{j}\,\mathrm{Sym}(w)$, where $w \in \mathcal{E}'(\boldsymbol{M}^{\times(k+\ell-2j)})$

[7] Here we use $\langle \delta^k \mathcal{H}[\overline{f}]^*/\delta f^k, u\rangle = \langle \delta^k \mathcal{H}[\overline{f}]/\delta \overline{f}^k, \overline{u}\rangle^*$.

is defined by $w(f \otimes g) = (W_{\boldsymbol{M}}^{\otimes j} v_g)(u_f)$ for $f \in \mathcal{E}(\boldsymbol{M}^{\times(k-j)})$, $g \in \mathcal{E}(\boldsymbol{M}^{\times(\ell-j)})$, regarding $W_{\boldsymbol{M}}^{\otimes j}$ as a map $\mathcal{T}^{(j)}(\boldsymbol{M}) \to \mathcal{D}'(\boldsymbol{M}^{\times j})$ and denoting $u_f(\cdot) = u(f \otimes \cdot) \in \mathcal{T}^{(j)}(\boldsymbol{M})$, $v_g(\cdot) = v(g \otimes \cdot) \in \mathcal{T}^{(j)}(\boldsymbol{M})$. The microlocal conditions on $W_{\boldsymbol{M}}$ and $\mathcal{T}^{(\bullet)}(\boldsymbol{M})$ ensure that all this is well-defined. Lastly, combining the field equation with Wick's formula gives $\mathcal{G}_{\boldsymbol{M}}[f + P_{\boldsymbol{M}} h] = \mathcal{G}_{\boldsymbol{M}}[f]$ and therefore, taking one functional derivative in h and the rest in f, yields the relations $:\Phi^{\otimes(k+1)}:_{\boldsymbol{M}}(w) = 0$ for any w in the closure of $\mathrm{span}\{\mathrm{Sym}(u \otimes P_{\boldsymbol{M}} v) : u \in \mathcal{T}^{(k)}(\boldsymbol{M}),\ v \in \mathcal{T}^{(1)}(\boldsymbol{M})\} \subset \mathcal{T}^{(k+1)}(\boldsymbol{M})$ for $k \in \mathbb{N}_0$ ($\mathcal{T}^{(0)}(\boldsymbol{M}) = \mathbb{C}$ by convention). The generating function evidently provides a very compact formulation of these relations and permits efficient computation with them.

Returning to the definition of $\mathcal{W}$ as a functor, to each morphism $\psi : \boldsymbol{M} \to \boldsymbol{N}$ in Loc, there is a corresponding $\mathcal{W}(\psi) : \mathcal{W}(\boldsymbol{M}) \to \mathcal{W}(\boldsymbol{N})$ which acts on generators by

$$\mathcal{W}(\psi)\mathcal{G}_{\boldsymbol{M}}[f] = \mathcal{G}_{\boldsymbol{N}}[\psi_* f] e^{(W_{\boldsymbol{M}}(f,f) - W_{\boldsymbol{N}}(\psi_* f, \psi_* f))/2} \tag{14}$$

and extends to an Alg-morphism (cf. [24, Sect. 3]) ultimately because $P_{\boldsymbol{N}} \psi_* f = \psi_* P_{\boldsymbol{M}} f$ for $f \in \mathcal{T}^{(1)}(\boldsymbol{M})$. Although $\mathcal{W}$ depends on the choice of $W_{\boldsymbol{M}}$'s, different choices result in equivalent theories. For our purposes we assume without loss that $W_{\lambda \boldsymbol{M}}(f, g) = \lambda^6 W_{\boldsymbol{M}}(f, f')$ for all $f, f' \in \mathcal{D}(\boldsymbol{M})$, $\lambda \in \mathbb{R}^+$. (This is consistent with the commutator condition because $\Box_{\lambda \boldsymbol{M}} = \lambda^{-2} \Box_{\boldsymbol{M}}$ and $d\mathrm{vol}_{\lambda \boldsymbol{M}} = \lambda^4 d\mathrm{vol}_{\boldsymbol{M}}$, giving $E^{\pm}_{\lambda \boldsymbol{M}} f = \lambda^2 E^{\pm}_{\boldsymbol{M}} f$ and $E^{\pm}_{\lambda \boldsymbol{M}}(f, f') = \lambda^6 E^{\pm}_{\boldsymbol{M}}(f, f')$.)

Covariance under rigid scaling We now show that $\mathcal{W}$ is $\mathbb{R}^+$-covariant under rigid scaling, for any $\xi \in \mathbb{R}$, by exhibiting natural isomorphisms $\eta(\lambda) : \mathcal{W} \stackrel{.}{\to} {}^{\lambda}\mathcal{W}$ for each $\lambda \in \mathbb{R}^+$, with components defined as

$$\eta(\lambda)_{\boldsymbol{M}} :\Phi^{\otimes k}_{\boldsymbol{M}}:(u) = \lambda^{-3k} :\Phi^{\otimes k}_{\lambda \boldsymbol{M}}:(u), \qquad (u \in \mathcal{T}^{(k)}(\boldsymbol{M}),\ \boldsymbol{M} \in \mathsf{Loc}).$$

Equivalently, $\eta(\lambda)_{\boldsymbol{M}} \mathcal{G}_{\boldsymbol{M}}[f] = \mathcal{G}_{\lambda \boldsymbol{M}}[\lambda^{-3} f]$, in which form compatibility with the relations may be verified easily. Hollands and Wald studied these maps in [24, Sect. 4.3] (notation differs) and showed that they are Alg-isomorphisms. Naturality was not proved in [24] but is easily checked: if $\psi : \boldsymbol{M} \to \boldsymbol{N}$ then

$$\begin{aligned} \eta(\lambda)_{\boldsymbol{N}} \mathcal{W}(\psi) \mathcal{G}_{\boldsymbol{M}}[f] &= \eta(\lambda)_{\boldsymbol{N}} \left(\mathcal{G}_{\boldsymbol{N}}[\psi_* f] e^{(W_{\boldsymbol{M}}(f,f) - W_{\boldsymbol{N}}(\psi_* f, \psi_* f))/2} \right) \\ &= \mathcal{G}_{\lambda \boldsymbol{N}}[\lambda^{-3} \psi_* f] e^{(W_{\boldsymbol{M}}(f,f) - W_{\boldsymbol{N}}(\psi_* f, \psi_* f))/2} \\ &= \mathcal{W}({}^{\lambda}\psi) \eta(\lambda)_{\boldsymbol{M}} \mathcal{G}_{\boldsymbol{M}}[f], \end{aligned}$$

using $W_{\lambda \boldsymbol{M}} = \lambda^6 W_{\boldsymbol{M}}$. This proves that $\mathcal{W}$ is $\mathbb{R}^+$-covariant. It is clear that $\eta(\lambda' \lambda)_{\boldsymbol{M}} = \eta(\lambda')_{\lambda \boldsymbol{M}} \eta(\lambda)_{\boldsymbol{M}}$, so the corresponding 2-cocycle takes the form (id, ϕ) where $\phi : \mathbb{R}^+ \to \mathrm{Aut}(\mathrm{Aut}(\mathcal{W}))$ remains to be determined.

For illustrative purposes, we restrict to the action of ϕ on a subgroup of $\mathrm{Aut}(\mathcal{W})$ which—on the basis of an analysis of the $\xi = 0$ unextended theory [15]—is expected to constitute all 'regular' gauge transformations. In the case $\xi \neq 0$, this subgroup is a $\mathbb{Z}_2$, with action defined by $\sigma_{\boldsymbol{M}} \mathcal{G}_{\boldsymbol{M}}[f] = \mathcal{G}_{\boldsymbol{M}}[\sigma f]$ ($\sigma = \pm 1$), while if $\xi = 0$ it is the nonabelian semidirect product $\mathbb{Z}_2 \ltimes \mathbb{R}$, with group product $(\sigma', \mu')(\sigma, \mu) = (\sigma' \sigma, \mu' \sigma + \mu)$ and action specified by

$$(\sigma, \mu)_{\boldsymbol{M}} \mathcal{G}_{\boldsymbol{M}}[f] = \mathcal{G}_{\boldsymbol{M}}[\sigma f] e^{i \mu \int_{\boldsymbol{M}} f d\mathrm{vol}_{\boldsymbol{M}}}. \tag{15}$$

Here, μ has dimensions of inverse length, like Φ. One may treat the two cases together by restricting to $\mu = 0$ if $\xi \neq 0$. Noting that

$$\eta(\lambda)_{\boldsymbol{M}}(\sigma,\mu)_{\boldsymbol{M}}\mathcal{G}_{\boldsymbol{M}}[f] = \eta(\lambda)_{\boldsymbol{M}}\mathcal{G}_{\boldsymbol{M}}[\sigma f]e^{i\mu\int f d\mathrm{vol}_{\boldsymbol{M}}} = \mathcal{G}_{\boldsymbol{M}}[\sigma f/\lambda^3]e^{i\frac{\mu}{\lambda}\int f/\lambda^3 d\mathrm{vol}_{\lambda\boldsymbol{M}}}$$
$$= (\sigma,\mu/\lambda)_{\lambda\boldsymbol{M}}\eta(\lambda)_{\boldsymbol{M}}\mathcal{G}_{\boldsymbol{M}}[f],$$

we have $\phi((\sigma,\mu)) = (\sigma,\mu/\lambda)$, which is consistent with the dimensions of μ. Thus, the 2-cocycle for rigid scaling is nontrivial for minimal coupling $\xi = 0$, and (at least its restriction to the regular subgroup) is trivial for $\xi \neq 0$.

Action on local Wick powers Scaling induces a group action on $\mathrm{Fld}(\mathcal{W})$ because $\mathcal{D}$ is also $\mathbb{R}^+$-covariant, implemented by $\lambda \mapsto \zeta^{(\alpha)}(\lambda)$, where $\alpha \in \mathbb{R}$ and $\zeta(\lambda)^{(\alpha)}_{\boldsymbol{M}} f = \lambda^{-4\alpha} f$ ($\mathcal{D}(\boldsymbol{M}) = \mathcal{D}(\lambda\boldsymbol{M})$, because the manifolds coincide). One may check that the corresponding cocycle is trivial for all α; we take $\alpha = 1$, so fields transform as densities of weight zero, and now drop the superscript α.

As suggested by the notation, the generators $:\Phi^{\otimes k}:_{\boldsymbol{M}}(u) \in \mathcal{W}(\boldsymbol{M})$ are (distributionally) smeared k-multilocal fields, Wick ordered with respect to $W_{\boldsymbol{M}}$.[8] Owing to (14), they do not transform covariantly for $k > 1$, because there is no choice of $W_{\boldsymbol{M}}$ such that $(\psi\times\psi)^* W_{\boldsymbol{N}} = W_{\boldsymbol{M}}$ for all $\psi : \boldsymbol{M} \to \boldsymbol{N}$. However, locally covariant Wick powers can be defined as follows. First, let $H_{\boldsymbol{M}}$ be the local Hadamard bidistribution, defined near the diagonal in $\boldsymbol{M}\times\boldsymbol{M}$ by

$$H_{\boldsymbol{M}}(p,q) = \frac{U_{\boldsymbol{M}}(p,q)}{4\pi^2\sigma_{\boldsymbol{M}+}(p,q)} + V_{\boldsymbol{M}}(p,q)\log(\sigma_{\boldsymbol{M}+}(p,q)/\ell^2)$$

where ℓ is a fixed length scale, common to all spacetimes, and $\sigma_{\boldsymbol{M}}(p,p')$ is the signed squared geodesic separation of p and p', with a positive sign for spacelike separation. The subscript + indicates that $f(\sigma_{\boldsymbol{M}+}(p,q)) = \lim_{\epsilon\to 0+} f(\sigma_{\boldsymbol{M}}(p,q) + 2i\epsilon(T_{\boldsymbol{M}}(p) - T_{\boldsymbol{M}}(q)) + \epsilon^2)$, where $T_{\boldsymbol{M}}$ increases to the future; $U_{\boldsymbol{M}}$ and $V_{\boldsymbol{M}}$ are smooth, and are fixed by requiring $U_{\boldsymbol{M}}(p,p) = 1$ and $(P_{\boldsymbol{M}}\otimes 1)H_{\boldsymbol{M}}(p,q) = O(\log(\sigma_{\boldsymbol{M}}(p,q)))$. At the diagonal, $W_{\boldsymbol{M}} - H_{\boldsymbol{M}}$ is continuous and $V_{\boldsymbol{M}}$ is a multiple of the Ricci scalar: $V_{\boldsymbol{M}}(p,p) = (6\xi - 1)R_{\boldsymbol{M}}|_p/(96\pi^2)$ (see, e.g. [10]).

With $H_{\boldsymbol{M}}$ so defined, set $\mathcal{H}_{\boldsymbol{M}}[f] = \mathcal{G}_{\boldsymbol{M}}[f]e^{(H_{\boldsymbol{M}}(f,f)-W_{\boldsymbol{M}}(f,f))/2}$ on f of sufficiently small support that $H_{\boldsymbol{M}}$ is defined on $\mathrm{supp}\, f \times \mathrm{supp}\, f$. Then

$$\Phi^k_{\boldsymbol{M}}(f) = \frac{1}{i^k}\left\langle \left.\frac{\delta^k\mathcal{H}_{\boldsymbol{M}}}{\delta h^k}\right|_{h=0}, f\delta^{(k)}_{\boldsymbol{M}}\right\rangle \tag{16}$$

defines a local k'th Wick power smeared against $f \in \mathcal{D}(\boldsymbol{M})$, where

$$(f\delta^{(k)}_{\boldsymbol{M}})(F) = \int_{\boldsymbol{M}} \rho_{\boldsymbol{M}}(p)^{-k} F(p,\ldots,p) f(p) d\mathrm{vol}_{\boldsymbol{M}}(p) \qquad (F \in \mathcal{E}(\boldsymbol{M}^{\times k}))$$

defines $f\delta^{(k)}_{\boldsymbol{M}} \in \mathcal{T}^{(k)}(\boldsymbol{M})$; here $\rho_{\boldsymbol{M}}$ is the density induced by $d\mathrm{vol}_{\boldsymbol{M}}$.

Under scaling, the transformed field obeys $(\lambda * \Phi^k)_{\boldsymbol{M}}(f) = \eta(\lambda)_{\boldsymbol{M}}\Phi^k_{\boldsymbol{M}}(\lambda^4 f)$, given our choice of ζ (see Theorem 8). Noting that

$$\eta(\lambda)_{\boldsymbol{M}}\mathcal{H}_{\boldsymbol{M}}[\lambda^4 f] = \mathcal{H}_{\lambda\boldsymbol{M}}[\lambda f]e^{\lambda^8 H_{\boldsymbol{M}}(f,f) - \lambda^2 H_{\lambda\boldsymbol{M}}(f,f)}$$
$$= \mathcal{H}_{\lambda\boldsymbol{M}}[\lambda f]e^{-\lambda^2\int\left(H_{\lambda\boldsymbol{M}}(p,q)-\lambda^{-2}H_{\boldsymbol{M}}(p,q)\right)f(p)f(q)d\mathrm{vol}^{\times 2}_{\lambda\boldsymbol{M}}(p,q)}$$

[8] Indeed, if $W_{\boldsymbol{M}}$ is of positive type, $W_{\boldsymbol{M}}(\overline{f}, f) \geq 0$ for all $f \in \mathcal{D}(\boldsymbol{M})$, one can define a state on $\mathcal{W}(\boldsymbol{M})$ in which all such elements have vanishing expectation value.

and using (16) together with the observations that $\lambda^4 f\delta^{(k)}_{\boldsymbol{M}} = \lambda^{4k} f\delta^{(k)}_{\lambda\boldsymbol{M}}$ and $H_{\lambda\boldsymbol{M}}(p,p) - \lambda^{-2}H_{\boldsymbol{M}}(p,p) = V_{\lambda\boldsymbol{M}}(p,p)\log\lambda^2$, a short calculation gives

$$\lambda * \Phi^k = \lambda^k \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{k!}{j!(k-2j)!}\left(\frac{6\xi - 1}{96\pi^2}\right)^j (\log\lambda^2)^j \mathsf{R}^j \Phi^{k-2j}, \tag{17}$$

where $\mathsf{R}^j\Phi^k \in \mathrm{Fld}(\mathcal{W})$ is the field $(\mathsf{R}^j\Phi^k)_{\boldsymbol{M}}(f) = \Phi^k_{\boldsymbol{M}}(R^j_{\boldsymbol{M}} f)$. Aside from the special cases $k = 1$ or $\xi = 1/6$, in which $\lambda * \Phi^k = \lambda^k\Phi^k$, all Wick powers obey 'almost homogeneous scaling' [24], and each Φ^k ($k \geq 2$) belongs to a $\lfloor k/2 \rfloor$-dimensional indecomposable (and reducible) $\mathbb{R}^+$-multiplet. Wick powers can be redefined within certain parameters [24], but homogeneous scaling cannot be regained: for example, it is possible to redefine Φ^2 by adding a fixed multiple of the Ricci scalar, but this still transforms inhomogeneously. We emphasise that, nonetheless, the theory $\mathcal{W}$ has rigid scale covariance for all $\xi \in \mathbb{R}$.

The above discussion can be compared with [35], which considered theories defined on a category CLoc that admits conformal isometries as morphisms. Only the $\xi = 1/6$ conformally coupled version of $\mathcal{W}$ is defined on CLoc and only locally conformally covariant fields can be discussed in that setting (these include Wick powers, related to those given above within the allowed renormalization freedoms). Our approach allows us to examine a broader class of theories that are scale covariant alongside theories that are not. By including the mass-squared parameter into the background category one can even discuss theories with mass (here the background objects are pairs $(\boldsymbol{M}, m^2)$ and $\mathbb{R}^+$ acts by $\mathcal{R}(\lambda)(\boldsymbol{M}, m^2) = (\lambda\boldsymbol{M}, m^2/\lambda^2)$). Elsewhere, it is hoped to explore the Stückelberg–Petermann renormalization group [3] in our framework.

Summarising, this example demonstrates the need for a cohomological description of G-covariance using nonabelian coefficients, the possibility of a nontrivial action of the group G ($\mathbb{R}^+$ for us) on the global gauge group and the possibility that fields can arise as indecomposable (but reducible) multiplets.

4. An Analogue of the Coleman–Mandula Theorem

4.1. Hypotheses, statement of main result and consequences. The purpose of this section is to prove Theorem 11, which shows that any theory $\mathcal{A} : \mathsf{FLoc} \to \mathsf{Phys}$ obeying mild conditions is covariant with respect to the universal covering group $\mathcal{S}$ of the restricted Lorentz group $\mathcal{L}_0$ (i.e., $\mathcal{S} \cong \mathrm{SL}(2,\mathbb{C})$ in 4 spacetime dimensions) and has trivial cohomology class. Accordingly, the Lorentz and internal symmetry groups do not mix, and the fields appear in $\mathcal{S}$ multiplets (if $\mathsf{Phys} = \mathsf{Alg}$, for example). Further consequences are discussed below.

To start, let us note that if $\mathcal{B} : \mathsf{Loc} \to \mathsf{Alg}$, then $\mathcal{A} := \mathcal{B} \circ \mathcal{F}_{\mathsf{L}} : \mathsf{FLoc} \to \mathsf{Alg}$ is certainly $\mathcal{L}_0$-covariant, because $\mathcal{F}_{\mathsf{L}}({}^{\Lambda}\mathcal{M}) = \mathcal{F}_{\mathsf{L}}(\mathcal{M})$ and $\mathcal{F}_{\mathsf{L}}({}^{\Lambda}\psi) = \mathcal{F}_{\mathsf{L}}(\psi)$ for all $\mathcal{M} \in \mathsf{FLoc}$ and all $\psi : \mathcal{M} \to \mathcal{N}$. Thus ${}^{\Lambda}\mathcal{A} = \mathcal{A}$ for all $\Lambda \in \mathcal{L}_0$, so the $\mathcal{L}_0$-covariance is implemented by $\Lambda \mapsto \mathrm{id}_{\mathcal{A}}$. The corresponding 2-cocycle is obviously trivial, and one obtains in a similar way that $\mathcal{A}$ is $\mathcal{S}$-covariant with trivial 2-cocycle. We have already shown that any theory $\mathcal{A} : \mathsf{SpinLoc} \to \mathsf{Alg}$ is $\mathcal{S}$-covariant with trivial 2-cocycle.[9] The purpose of Theorem 11 is not to describe these cases as such, but rather to show why *all*

[9] It follows that $\mathcal{A} \circ \mathcal{F}_{\mathsf{S}}$ is $\mathcal{S}$-covariant with neutral cocycle $(1, \phi)$, where ϕ is trivial on $\mathcal{F}^*_{\mathsf{S}}(\mathrm{Aut}(\mathcal{A}))$, which could *a priori* be a proper subgroup of $\mathrm{Aut}(\mathcal{A} \circ \mathcal{F}_{\mathsf{S}})$.

theories on FLoc obeying our conditions, however constructed, have a trivial cocycle for a common reason. We now proceed to assemble the hypotheses and concepts required in Theorem 11.

Timeslice property Given $\mathfrak{M} = (\mathcal{M}, e) \in \mathsf{FLoc}$, we will say that a set $\Sigma \subset \mathcal{M}$ is a Cauchy surface if it is intersected exactly once by every e-timelike curve. A morphism $\psi : \mathfrak{M} \to \mathfrak{M}'$ is said to be Cauchy if the image of ψ contains a Cauchy surface of $\mathfrak{M}'$. Thus a FLoc-morphism ψ is Cauchy if and only if $\mathcal{F}_{\mathsf{L}}(\psi)$ is Cauchy in Loc according to the terminology of [19]. The theory $\mathcal{A}$ has the *timeslice property* if $\mathcal{A}(\psi)$ is an isomorphism for all Cauchy ψ.

Relative Cauchy evolution & dynamical local Lorentz invariance Relative Cauchy evolution measures the response of the dynamics of a theory to a variation in the background structures. In FLoc, variations of $\mathfrak{M} = (\mathcal{M}, e)$ are parametrized by a smooth function $T \in C^\infty_{\mathrm{tc}}(\mathcal{M}; \mathrm{GL}^+(n; \mathbb{R}))$ where the subscript indicates that $\operatorname{supp} T$ (the closure of the subset of $\mathcal{M}$ on which T differs from the identity) is time-compact. The varied coframe is Te, where $(Te)^\mu|_p = T^\mu{}_\nu(p) e^\nu|_p$; we restrict to those T for which $\mathfrak{M}[T] := (\mathcal{M}, Te)$ is an object of FLoc. Coframe variations include, but go beyond, the metric variations studied in [4,19,20]—they can also be used to detect whether a theory is sensitive to local Lorentz transformations. (Frame variations are required in describing relative Cauchy evolution in the Dirac case [12,39] but in an auxiliary role, whereas here they are primary.)

Let $\mathcal{M}^\pm = I^\pm_{\mathfrak{M}}(\Sigma^\pm)$ where $\Sigma^\pm$ are smooth spacelike Cauchy surfaces obeying $\operatorname{supp} T \subset I^+_{\mathfrak{M}}(\Sigma^-) \cap I^-_{\mathfrak{M}}(\Sigma^+)$. Then $\mathfrak{M}^\pm = (\mathcal{M}^\pm, e|_{\mathcal{M}^\pm})$ are objects of FLoc and the subset inclusions of $\mathcal{M}^\pm$ in $\mathcal{M}$ induce Cauchy morphisms $\iota^\pm : \mathfrak{M}^\pm \to \mathfrak{M}$ and $\iota^\pm[T] : \mathfrak{M}^\pm \to \mathfrak{M}[T]$. The *relative Cauchy evolution* $\mathrm{rce}_{\mathfrak{M}}[T]$ is defined by

$$\mathrm{rce}_{\mathfrak{M}}[T] = \mathcal{A}(\iota^-)\mathcal{A}(\iota^-[T])^{-1}\mathcal{A}(\iota^+[T])\mathcal{A}(\iota^+)^{-1}$$

and is clearly an automorphism of $\mathcal{A}(\mathfrak{M})$, assuming $\mathcal{A}$ has the timeslice property. The specific choice of frame should be irrelevant in physical theories, motivating:

Definition 10. A theory $\mathcal{A} : \mathsf{FLoc} \to \mathsf{Phys}$ with the timeslice property satisfies dynamical local Lorentz invariance if $\mathrm{rce}_{\mathfrak{M}}[\tilde{\Lambda}] = \mathrm{id}$ for all $\mathfrak{M} \in \mathsf{FLoc}$ and all $\tilde{\Lambda} \in C^\infty_{\mathrm{tc}}(\mathcal{M}; \mathcal{L}_0)$ that are null-homotopic relative to the complement of a time-compact subset of $\mathcal{M}$.

This condition holds in any theory induced from Loc or $\mathsf{SpinLoc}$ of the form $\mathcal{A} = \mathcal{B} \circ \mathcal{F}_{\mathsf{L}}$ or $\mathcal{A} = \mathcal{C} \circ \mathcal{F}_{\mathsf{S}}$.[10] Note that the restriction to null-homotopic $\tilde{\Lambda}$ is a conservative assumption; a stronger definition that dropped the null-homotopy condition would rule out theories with non-integer spin fields.[11]

Additivity The theory $\mathcal{A}$ is said to be additive if each $\mathcal{A}(\mathfrak{M})$ can be built from knowledge of the theory on suitable subregions of $\mathfrak{M}$. To make this precise, note first that if $\mathfrak{M} = (\mathcal{M}, e)$ and O is an open e-causally convex subset of $\mathcal{M}$, then $\mathfrak{M}|_O := (O, e|_O)$ defines the FLoc-object corresponding to O as a spacetime in its own right, and that the inclusion of O in $\mathcal{M}$ induces a FLoc-morphism $\iota_{\mathfrak{M};O} : \mathfrak{M}|_O \to \mathfrak{M}$. Our additivity condition requires that the morphisms $\mathcal{A}(\iota_{\mathfrak{M};D})$ are jointly epic as D runs over the set of truncated multi-diamonds (defined below) in $\boldsymbol{M}$: that is, if $\alpha \circ \mathcal{A}(\iota_{\mathfrak{M};D}) = \beta \circ \mathcal{A}(\iota_{\mathfrak{M};D})$ for all

[10] The Loc case is trivial, because $\mathcal{F}_{\mathsf{L}}(\mathfrak{M}[T]) = \mathcal{F}_{\mathsf{L}}(\mathfrak{M})$; $\mathsf{SpinLoc}$ needs a short calculation.

[11] The need to consider homotopy properties of framings in relation to relative Cauchy evolution was noted by Ferguson [12].

truncated multi-diamonds D, then $\alpha = \beta$. This differs slightly from the definition used in [15, 19] but follows from it if (as is true for $\mathsf{Phys} = \mathsf{Alg}$) Phys has unions and equalizers [15, Lem 2.5].[12] A *truncated multi-diamond* is a subset of the form $\mathcal{N} \cap D_{\mathbf{M}}(B)$ where $\mathcal{N}$ is an open globally hyperbolic neighbourhood of Cauchy surface Σ in $\mathbf{M}$, while the *base* B is a finite union of disjoint subsets of Σ each of which is an open ball in local coordinates, and is called a *Cauchy multi-ball*. Images of Cauchy multi-balls under $(\mathsf{F})\mathsf{Loc}$ morphisms are again Cauchy multi-balls. (See Def. 2.5 and the subsequent discussion in [19].)

Given these definitions, our main result can be stated as follows.

Theorem 11. *In spacetime dimension $n \geq 2$, suppose $\mathcal{A} : \mathsf{FLoc} \to \mathsf{Phys}$ obeys the timeslice axiom, dynamical local Lorentz invariance and additivity. Then $\mathcal{A}$ is $\mathcal{S}$-covariant with trivial cocycle (and hence trivial cohomology class).*

Before giving the proof we make some remarks and draw out some consequences. First, as discussed in the introduction, Theorem 11 is an analogue of the Coleman–Mandula Theorem [7] insofar as it is based on dynamics (specifically, the timeslice property and dynamical local Lorentz invariance), rather than on a group theoretic analysis such as [27, 30, 32]. However, we re-emphasize that our result is not a direct generalization of the Coleman–Mandula theorem in either its statement or its method of proof. It is also worth noting that the proof of Theorem 11 does not utilize special properties of Minkowski spacetime, or of the theory $\mathcal{A}$ restricted to Minkowski spacetime. In this, it differs from results such as the spin-statistics connection [41].

Second, triviality of the cohomology class implies that the corresponding extended symmetry group is a direct product $E = \mathrm{Aut}(\mathcal{A}) \times \mathcal{S}$. The single-component fields $\mathrm{Fld}(\mathcal{A})$ therefore form multiplets under the action of E, and the restrictions of this action to $\mathrm{Aut}(\mathcal{A})$ or $\mathcal{S}$ are also true representations. Thus fields arise in $\mathcal{S}$-multiplets, just as in Minkowski spacetime. By Corollary 9, inequivalent irreducible representations of $\mathcal{S}$ (or indeed of the gauge group $\mathrm{Aut}(\mathcal{A})$) cannot be mixed by the action of E, so finite-dimensional multiplets of different spinor-tensor type do not mix.

Third, the proof of Theorem 11 explicitly constructs an implementation of the $\mathcal{S}$-covariance. In Minkowski spacetime, this can be connected to the standard action of the Lorentz group in Wightman theory—see Sect. 4.3.

Fourth, any $\mathcal{S}$-covariant theory is also $\mathcal{L}_0$-covariant with an implementation given by $\Lambda \mapsto \eta(\Lambda) = \zeta(S_\Lambda)$, where $\Lambda \mapsto S_\Lambda$ is any section of the covering homomorphism $\pi : \mathcal{S} \to \mathcal{L}_0$ with $S_{\mathbf{1}} = 1$. The corresponding cocycle is easily calculated, using triviality of that induced by ζ, and is $(\zeta \circ z, 1)$, where $z : \mathcal{L}_0 \times \mathcal{L}_0 \to \ker \pi$ is given as $z(\Lambda', \Lambda) = S_{\Lambda'} S_\Lambda S_{\Lambda'\Lambda}^{-1}$. The restriction of ζ to $\ker \pi$ is therefore of interest.

Lemma 12. *ζ restricts to a homomorphism from* $\ker \pi$ *to the centre* $Z(\mathrm{Aut}(\mathcal{A}))$.

Proof. For $S \in \ker \pi$, ${}^S\mathbf{M} = \mathbf{M}$ for each $\mathbf{M}$ and so $\zeta(S) \in \mathrm{Aut}(\mathcal{A})$. Triviality of the $\mathcal{S}$ cocycle induced by ζ (cf. (6)) implies that $\zeta|_{\ker \pi}$ is a homomorphism and that $\zeta(S)\alpha = \alpha\zeta(S)$ for all $\alpha \in \mathrm{Aut}(\mathcal{A})$, $S \in \ker \pi$. $\square$

The kernel of π is the homotopy group $\pi_1(\mathcal{L}_0)$. In spacetime dimensions $n \geq 4$, $\ker \pi \cong \mathbb{Z}_2$, and $\zeta(-1)$ is thus an involutive, central element of $\mathrm{Aut}(\mathcal{A})$, while in $n = 3$, $\ker \pi$ is the infinite cyclic group, and in $n = 2$, it is trivial. The extended group corresponding to $\mathcal{L}_0$-covariance is a quotient of $\mathrm{Aut}(\mathcal{A}) \times \mathcal{S}$; for example, if $n \geq 4$, it is $(\mathrm{Aut}(\mathcal{A}) \times$

[12] There is a typographical error in the proof of [15, Lem 2.5]; the calculation in the penultimate line should end with $h \circ m$, not m.

$\mathcal{S})/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ is generated by $(\zeta(-1), -1)$. As all fields transform in true $\mathcal{L}_0$-representations if $(\zeta \circ z, 1)$ is trivial, one has:

Corollary 13. *If $n \geq 3$, let $\mathcal{A}$ obey the conditions of Theorem 11. A necessary condition for* $\mathrm{Fld}(\mathcal{A})$ *to contain multiplets of noninteger spin is that* $\mathrm{Aut}(\mathcal{A})$ *carries a nontrivial homomorphic image of $\pi_1(\mathcal{L}_0)$ (induced by ζ). In particular, $\zeta(-1)$ must be a nontrivial involutive central element in dimension $n \geq 4$.*

The structure of $Z(\mathrm{Aut}(\mathcal{A}))$ therefore constrains the possible spins of fields associated with $\mathcal{A}$. For example, any theory described by algebras of *observables* (as opposed to possibly unobservable quantities) has trivial global gauge group and thus can only support multiplets of integer spin. We will return elsewhere [13] to the role of the *univalence* $\zeta(-1)$ in the spin-statistics connection (see [17,18] for brief accounts). Finally, it has already been noted that all theories on FLoc of the form $\mathcal{A} = \mathcal{B} \circ \mathcal{F}_{\mathsf{L}}$ are $\mathcal{L}_0$-covariant with trivial cocycle. Accordingly the fields in $\mathrm{Fld}(\mathcal{A})$ transform under true $\mathcal{L}_0$-representations, proving that no theory with noninteger spin can be constructed on Loc.

4.2. Proof of Theorem 11. The proof has three parts: (a) for $\mathcal{M} \in \mathsf{FLoc}$, $S \in \mathcal{S}$, we construct isomorphisms $\zeta_{\mathcal{M}}(S) : \mathcal{A}(\mathcal{M}) \to \mathcal{A}({}^S\mathcal{M})$; (b) we prove that the $\zeta_{\mathcal{M}}(S)$ cohere to form natural isomorphisms and therefore implement $\mathcal{S}$-covariance of $\mathcal{A}$; (c) we compute the corresponding 2-cocycle. Additivity is used in part (b) for reasons discussed below, while dynamical local Lorentz invariance is used to show that $\zeta_{\mathcal{M}}(S)$ is independent of various choices made in its construction, which is important in (b) and (c). Throughout, we use the fact that elements of $\mathcal{S}$ can be regarded as homotopy equivalence classes of curves in $\mathcal{L}_0$ with a base-point at the identity I. We now take these parts in turn.

(a) Construction of $\zeta_{\mathcal{M}}(S)$ Fix $\mathcal{M} = (\mathcal{M}, e)$ and $S \in \mathcal{S}$. Choose $\tilde{\Lambda} \in C^\infty(\mathcal{M}; \mathcal{L}_0)$ obeying $\tilde{\Lambda} \equiv I$ on $J^-_{\mathcal{M}}(\Sigma^-)$ and $\tilde{\Lambda} \equiv \Lambda$ on $J^+_{\mathcal{M}}(\Sigma^+)$, where $\Sigma^\pm$ are smooth spacelike Cauchy surfaces with $\Sigma^\pm \subset I^\pm_{\mathcal{M}}(\Sigma^\mp)$; it is required that $\tilde{\Lambda}$ has homotopy class S relative to $J^+_{\mathcal{M}}(\Sigma^+) \cup J^-_{\mathcal{M}}(\Sigma^-)$ (in every component of $\mathcal{M}$).[13] Next, define $\tilde{\mathcal{M}} = (\mathcal{M}, \tilde{\Lambda} e)$ (abusing notation, we will sometimes write $\tilde{\mathcal{M}} = {}^{\tilde{\Lambda}}\mathcal{M}$) and also $\mathcal{M}^\pm = (\mathcal{M}^\pm, e|_{\mathcal{M}^\pm})$ where $\mathcal{M}^\pm = I^\pm_{\mathcal{M}}(\Sigma^\pm)$. The obvious Cauchy morphisms induced by subset inclusions

$$\mathcal{M} \xleftarrow{\iota^-} \mathcal{M}^- \xrightarrow{\tilde{\iota}^-} \tilde{\mathcal{M}} \xleftarrow{\tilde{\iota}^+} {}^{\Lambda}\mathcal{M}^+ \xrightarrow{\iota^+} {}^{\Lambda}\mathcal{M} \tag{18}$$

(see Fig. 1) induce an isomorphism $\zeta(\tilde{\Lambda}) : \mathcal{A}(\mathcal{M}) \to \mathcal{A}({}^{\Lambda}\mathcal{M})$,

$$\zeta(\tilde{\Lambda}) = \mathcal{A}(\iota^+)\mathcal{A}(\tilde{\iota}^+)^{-1}\mathcal{A}(\tilde{\iota}^-)\mathcal{A}(\iota^-)^{-1} \tag{19}$$

by the timeslice property. We will describe $\zeta(\tilde{\Lambda})$ as being formed by 'chasing the arrows' in (18) from $\mathcal{M}$ to ${}^{\Lambda}\mathcal{M}$. Note that if $\mathcal{A} = \mathcal{B} \circ \mathcal{F}_{\mathsf{L}}$ then, because $\mathcal{F}_{\mathsf{L}}(\mathcal{M}) = \mathcal{F}_{\mathsf{L}}(\tilde{\mathcal{M}})$, we have $\iota^\pm = \tilde{\iota}^\pm$ (recall that each of these morphisms has an inclusion as its underlying map) and hence $\zeta(\tilde{\Lambda}) = \mathrm{id}_{\mathcal{B}(\mathcal{M})}$ for any $\tilde{\Lambda}$.

We now show that the construction of $\zeta(\tilde{\Lambda})$ is independent of the choice of $\Sigma^\pm$ and depends only on the homotopy class of $\tilde{\Lambda}$. Starting with the Cauchy surfaces, note that

[13] In each component, every timelike curve from the past of Σ^- to the future of Σ^+ induces a curve connecting I to Λ in $\mathcal{L}_0$, and these curves must have common homotopy type.

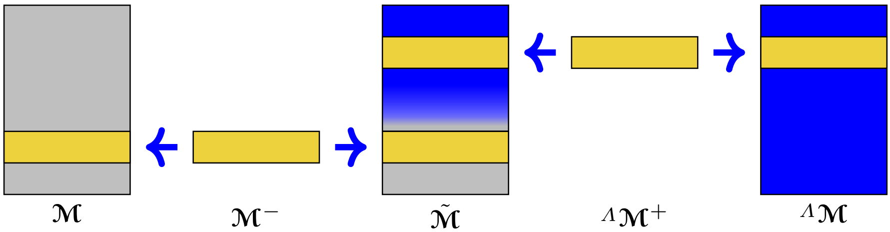

Fig. 1. Diagram of spacetimes involved in constructing $\zeta(\tilde{\Lambda})$

whenever Σ and Σ' are smooth spacelike Cauchy surfaces, there is a smooth spacelike Cauchy surface Σ'' in their common future (or past). Hence it is enough to show that $\zeta(\tilde{\Lambda})$ is also obtained if smooth spacelike Cauchy surfaces $\hat{\Sigma}^\pm \subset I^\pm_{\mathfrak{M}}(\Sigma^\pm)$ are used in place of $\Sigma^\pm$, leaving $\tilde{\Lambda}$ unchanged. Defining $\hat{\mathfrak{M}}^\pm$ by analogy with $\mathfrak{M}^\pm$, the Cauchy morphisms of $\hat{\mathfrak{M}}^-$ into $\mathfrak{M}$ and $\tilde{\mathfrak{M}}$ factor via the Cauchy morphism $j^- : \hat{\mathfrak{M}}^- \to \mathfrak{M}^-$, i.e., $\hat{\iota}^- = \iota^- \circ j^-$, and $\tilde{\hat{\iota}}^- = \tilde{\iota}^- \circ j^-$. Thus

$$\mathcal{A}(\tilde{\hat{\iota}}^-)\mathcal{A}(\hat{\iota}^-)^{-1} = \mathcal{A}(\tilde{\iota}^-)\mathcal{A}(j^-)\mathcal{A}(j^-)^{-1}\mathcal{A}(\iota^-)^{-1} = \mathcal{A}(\tilde{\iota}^-)\mathcal{A}(\iota^-)^{-1}; \tag{20}$$

a similar argument applies to $\mathcal{A}(\iota^+)\mathcal{A}(\tilde{\iota}^+)^{-1}$ and establishes the required independence. Similarly, the isomorphism $\zeta(\tilde{\Lambda})$ is also unchanged if we replace $\mathfrak{M}^\pm$ by causally convex subsets thereof that contain Cauchy surfaces of $\mathfrak{M}$.

Next, let $\hat{\Lambda} \in C^\infty(\mathcal{M}; \mathcal{L}_0)$ obey the same conditions as $\tilde{\Lambda}$ (relative to a common choice of Cauchy surfaces $\Sigma^\pm$ without loss of generality), thus also having homotopy class S. Then we have the following diagram of Cauchy morphisms

$$\begin{tikzcd}
 & & \tilde{\mathfrak{M}} & & \\
\mathfrak{M} & \mathfrak{M}^- \arrow[l, "\iota^-"'] \arrow[ur, "\tilde{\iota}^-"] \arrow[dr, "\hat{\iota}^-"'] & & {}^{\Lambda}\mathfrak{M}^+ \arrow[ul, "\tilde{\iota}^+"'] \arrow[dl, "\hat{\iota}^+"] \arrow[r, "\iota^+"] & {}^{\Lambda}\mathfrak{M} \\
 & & \hat{\mathfrak{M}} & &
\end{tikzcd} \tag{21}$$

and the isomorphism $\zeta(\hat{\Lambda}) : \mathcal{A}(\mathfrak{M}) \to \mathcal{A}({}^{\Lambda}\mathfrak{M})$ is formed by chasing arrows along the lower branch from $\mathfrak{M}$ to ${}^{\Lambda}\mathfrak{M}$. Now, $\hat{\Lambda}\tilde{\Lambda}^{-1}$ acts trivially outside a timelike-compact set and is by assumption null-homotopic relative to the complement of the time-compact subset of $\mathcal{M}$ bounded by $\Sigma^\pm$. Dynamical local Lorentz invariance then implies that $\mathrm{rce}_{\tilde{\mathfrak{M}}}[\hat{\Lambda}\tilde{\Lambda}^{-1}]$ is trivial, so the diamond in (21) commutes and $\zeta(\tilde{\Lambda}) = \zeta(\hat{\Lambda})$. As the isomorphism depends only on the homotopy class S, it will henceforth be denoted $\zeta_{\mathfrak{M}}(S)$.

(b) Naturality of $\zeta(S)$ Let $\psi : \mathfrak{M} \to \mathfrak{N}$ be arbitrary. We must show that

$$\zeta_{\mathfrak{N}}(S) \circ \mathcal{A}(\psi) = \mathcal{A}({}^S\psi) \circ \zeta_{\mathfrak{M}}(S) \tag{22}$$

holds for the isomorphisms defined in part (a). The obstruction to a straightforward proof of (22) is that the interpolating spacetime $\tilde{\mathfrak{M}}$ used to construct $\zeta_{\mathfrak{M}}(S)$ (see Fig. 1) might not embed in an interpolating spacetime for the construction of $\zeta_{\mathfrak{N}}(S)$. Indeed, the function $\tilde{\Lambda} \in C^\infty(\mathfrak{M}; \mathcal{L}_0)$ might not be the pull-back of a function in $C^\infty(\mathfrak{N}; \mathcal{L}_0)$—for example, $\psi(\mathfrak{M})$ might have boundary points to which the push-forward $\psi_*\tilde{\Lambda}$ cannot

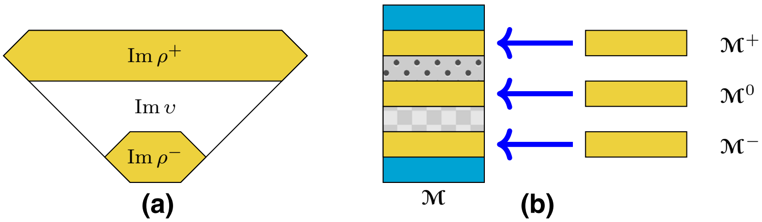

Fig. 2. a The arrangement of image regions $\operatorname{Im}\rho^\pm \subset \operatorname{Im}\upsilon \subset D_{\mathcal{M}}(\operatorname{Im}\rho^+)$ used in Lemma 14. **b** The spacetimes used to compute the cocycle. The map $\tilde{\Lambda}_T$ (resp. $\tilde{\Lambda}_S$) is locally constant outside the *chequered* (resp., *dotted*) region

be extended continuously. We circumvent this problem using an argument based on additivity.

A further definition is required: Given open subsets $R^\pm$ of a spacetime $\mathcal{M} \in \mathsf{FLoc}$, $\mu \in C^\infty(\mathcal{M}, [0, 1])$ is a *temporal mollifier* for the ordered pair (R^-, R^+) if there exist smooth spacelike Cauchy surfaces $\Sigma^\pm$ for $\mathcal{M}$ so that $R^\pm \subset I^\pm_{\mathcal{M}}(\Sigma^\pm)$, and μ vanishes identically on $J^-_{\mathcal{M}}(\Sigma^-)$ (hence on R^-) and is identically unity on $J^+_{\mathcal{M}}(\Sigma^+)$ (hence on R^+). Temporal mollifiers exist for (R^-, R^+) if and only $R^{+/-}$ lie to the future/past of a smooth spacelike Cauchy surface.

The proof of (22) falls into two parts: first, in Lemma 14 we show that it holds on subspacetimes of $\mathcal{M}$ provided suitable temporal mollifiers can be found; second, in Lemma 15, we show how such mollifiers may be constructed on a sufficiently large class of subspacetimes to establish naturality if $\mathcal{A}$ is additive.

Lemma 14. *Fix $S \in \mathcal{S}$ and also a smooth path $\sigma : [0, 1] \to \mathcal{S}$ with $\sigma(0) = \mathbf{1}$, $\sigma(1) = S$. (i) Let $\mathcal{M} \in \mathsf{FLoc}$ and suppose morphisms $\rho^\pm : \mathcal{R}^\pm \to \mathcal{M}$ and $\upsilon : \mathfrak{U} \to \mathcal{M}$ are given with image regions obeying $\operatorname{Im}\rho^\pm \subset \operatorname{Im}\upsilon \subset D_{\mathcal{M}}(\operatorname{Im}\rho^+)$ (see Fig. 2a). If μ is a temporal mollifier for $(\operatorname{Im}\rho^-, \operatorname{Im}\rho^+)$ in $\mathcal{M}$, then*

$$\zeta_{\mathcal{M}}(S)\mathcal{A}(\rho^-) = \mathcal{A}({}^S\rho^+)\mathcal{A}(\kappa^+)^{-1}\mathcal{A}(\kappa^-), \tag{23}$$

where the morphisms $\mathcal{R}^- \xrightarrow{\kappa^-} {}^{\upsilon^*(\sigma\circ\mu)}\mathfrak{U} \xleftarrow{\kappa^+} {}^S\mathcal{R}^+$, of which κ^+ is Cauchy, are uniquely determined by the requirement that $\mathcal{F}_{\mathsf{L}}(\upsilon) \circ \mathcal{F}_{\mathsf{L}}(\kappa^\pm) = \mathcal{F}_{\mathsf{L}}(\rho^\pm)$.

(ii) If, additionally, $\psi : \mathcal{M} \to \mathcal{N}$, suppose that a temporal mollifier ν exists for $(\operatorname{Im}\psi \circ \rho^-, \operatorname{Im}\psi \circ \rho^+)$ in $\mathcal{N}$ that obeys $\psi^*\nu = \mu$ on $\upsilon(\mathfrak{U})$. Then

$$\zeta_{\mathcal{N}}(S)\mathcal{A}(\psi)\mathcal{A}(\rho^-) = \mathcal{A}({}^S\psi)\zeta_{\mathcal{M}}(S)\mathcal{A}(\rho^-). \tag{24}$$

Proof. (i) Select smooth spacelike Cauchy surfaces $\Sigma^\pm$ in $\mathcal{M}$ so that μ vanishes to the past of Σ^- and is identically unity on the future of Σ^+, arranging also that $\operatorname{Im}\rho^\pm \subset I^\pm_{\mathcal{M}}(\Sigma^\pm)$. Using these Cauchy surfaces to define $\mathcal{M}^\pm$ as in part (a), and building the interpolating spacetime $\tilde{\mathcal{M}}$ using $\tilde{\Lambda} = \pi(\tilde{S})$ where $\tilde{S} = \sigma \circ \mu$, we construct $\zeta_{\mathcal{M}}(S)$ by chasing arrows from left to right along the top line of

$$\begin{tikzcd}
\mathcal{M} & \mathcal{M}^- \arrow[l, "c"'] \arrow[r, "c"] & \tilde{\mathcal{M}} & {}^S\mathcal{M}^+ \arrow[l, "c"'] \arrow[r, "c"] & {}^S\mathcal{M} \\
& \mathcal{R}^- \arrow[ul, "\rho^-"] \arrow[u, dashed] \arrow[r, dotted, "\kappa^-"'] & \tilde{\mathfrak{U}} \arrow[u, "\tilde{\upsilon}"] & {}^S\mathcal{R}^+ \arrow[u, dashed] \arrow[l, dotted, "c", "\kappa^+"'] \arrow[ur, "{}^S\rho^+"'] &
\end{tikzcd}$$

in which $\tilde{\mathfrak{U}} = {}^{\upsilon^*\tilde{S}}\mathfrak{U}$ and $\tilde{\upsilon} = {}^{\upsilon^*\tilde{S}}\upsilon$. We now establish the existence of the dashed and dotted morphisms and show that the diagram commutes, from which (23) follows by functoriality and the timeslice property. As Im $\rho^\pm \subset I^\pm_{\mathfrak{M}}(\Sigma^\pm)$, there are unique dashed morphisms as shown, making the two triangles commute, and inducing morphisms from $\mathfrak{R}^-$ and ${}^S\mathfrak{R}^+$ to $\tilde{\mathfrak{M}}$ via $\mathfrak{M}^-$ and ${}^S\mathfrak{M}^+$; these morphisms have the same underlying functions as $\rho^\pm$. The conditions on Im $\rho^\pm$ and Im υ entail that there are (unique) morphisms $\kappa^\pm$ making the squares commute and with κ^+ Cauchy, with underlying functions obeying $\upsilon \circ \kappa^\pm = \rho^\pm$; more formally we may write $\mathcal{F}_{\mathsf{L}}(\upsilon) \circ \mathcal{F}_{\mathsf{L}}(\kappa^\pm) = \mathcal{F}_{\mathsf{L}}(\rho^\pm)$, and this relation determines $\kappa^\pm$ uniquely because their codomain $\tilde{\mathfrak{U}}$ is fixed. Part (i) is thus proved.

(ii) We apply part (i) to $\psi \circ \rho^\pm : \mathfrak{R}^\pm \to \mathfrak{N}$, and $\psi \circ \upsilon : \mathfrak{U} \to \mathfrak{N}$ using $\pi(\sigma \circ \nu)$ to build an interpolating spacetime for the construction of $\zeta_{\mathfrak{N}}(S)$. Note that $(\psi \circ \upsilon)^*\nu = \upsilon^*\psi^*\nu = \upsilon^*\mu$, so

$$ {}^{(\psi\circ\upsilon)^*(\sigma\circ\nu)}\mathfrak{U} = {}^{\upsilon^*(\sigma\circ\mu)}\mathfrak{U} = \tilde{\mathfrak{U}}. $$

Therefore one has $\zeta_{\mathfrak{N}}(S)\mathcal{A}(\psi \circ \rho^-) = \mathcal{A}({}^S(\psi \circ \rho^+))\mathcal{A}(\kappa^+)^{-1}\mathcal{A}(\kappa^-)$ with the same morphisms $\kappa^\pm$ as in the original application of part (i), because those morphisms obviously satisfy the characterising equation $\mathcal{F}_{\mathsf{L}}(\psi \circ \upsilon) \circ \mathcal{F}_{\mathsf{L}}(\kappa^\pm) = \mathcal{F}_{\mathsf{L}}(\psi \circ \rho^\pm)$ and have the same codomain $\tilde{\mathfrak{U}}$. Combining this with (23) gives

$$ \zeta_{\mathfrak{N}}(S)\mathcal{A}(\psi)\mathcal{A}(\rho^-) = \mathcal{A}({}^S\psi)\mathcal{A}({}^S\rho^+)\mathcal{A}(\kappa^+)^{-1}\mathcal{A}(\kappa^-) = \mathcal{A}({}^S\psi)\zeta_{\mathfrak{M}}(S)\mathcal{A}(\rho^-). $$

□

The above circumstances can be achieved for sufficiently many ρ^- to form a jointly epic set of morphisms $\mathcal{A}(\rho^-)$.

Lemma 15. *Suppose $\psi : \mathfrak{M} \to \mathfrak{N}$ in* FLoc *and let $S \in \mathcal{S}$. For any truncated multi-diamond D of $\mathfrak{M}$, we have*

$$ \zeta_{\mathfrak{N}}(S)\mathcal{A}(\psi)\mathcal{A}(\iota_{\mathfrak{M};D}) = \mathcal{A}({}^S\psi)\zeta_{\mathfrak{M}}(S)\mathcal{A}(\iota_{\mathfrak{M};D}). \tag{25} $$

The additivity assumption on $\mathcal{A}$ entails that the $\mathcal{A}(\iota_{\mathfrak{M};D})$ are jointly epic as D runs over the truncated multi-diamonds. Therefore one has $\zeta_{\mathfrak{N}}(S)\mathcal{A}(\psi) = \mathcal{A}({}^S\psi)\zeta_{\mathfrak{M}}(S)$, and as $\psi : \mathfrak{M} \to \mathfrak{N}$ was arbitrary naturality of $\zeta(S)$ is established.

Proof of Lemma 15. Let D be based on a Cauchy multi-ball $B^- \subset \Sigma$, where Σ is a smooth spacelike Cauchy surface. By [2, Thm 1.2], we may find a Cauchy temporal function foliating $\mathcal{M}$ as $\mathbb{R} \times \Sigma$ so that D has base $\{0\} \times B^-$ and the e-metric splits orthogonally as $\beta \oplus -h_t$, where h_t is a smooth Riemannian metric on Σ for each $t \in \mathbb{R}$ and $\beta \in C^\infty(\mathcal{M})$ is nonnegative. The significance of this splitting is that each $\{t\} \times \Sigma$ is a smooth spacelike Cauchy surface and all curves $t \mapsto (t, \sigma)$ for fixed $\sigma \in \Sigma$ are timelike. This facilitates bounds on Cauchy developments, e.g., $D_{\mathfrak{M}}(\{t\} \times B) \subset \mathbb{R} \times B$.

The form of the e-metric allows us to choose another Cauchy multi-ball $\{0\} \times B^+$ containing the closure of $\{0\} \times B^-$ and $\epsilon > 0$ such that $\{0\} \times B^- \subset D_{\mathfrak{M}}(\{t\} \times B^+)$ for all $0 < t < \epsilon$ (cf. [16, Lem. 2.5]). Choosing $t^+ \in (0, \epsilon)$ and setting $t^- = 0$, we define truncated multi-diamonds

$$ R^\pm = \{(t, \sigma) \in D_{\mathfrak{M}}(\{t^\pm\} \times B^\pm) : |t - t^\pm| < t^+/3\} $$

based on the Cauchy multi-balls $\{t^\pm\} \times B^\pm$. Setting

$$ U = \{(t, \sigma) \in D_{\mathfrak{M}}(R^+) : -t^+/3 < t < 4t^+/3\}, $$

it is evident that $R^\pm \subset U \subset D_{\mathfrak{M}}(R^+)$. We may choose a temporal mollifier μ for (R^-, R^+) so that μ vanishes on $(-\infty, 4t^+/9] \times \Sigma$ and μ is unity on $[5t^+/9, \infty) \times \Sigma$.

Supposing that $\psi : \mathfrak{M} \to \mathfrak{N}$, we now construct a temporal mollifier ν for $(\psi(R^-), \psi(R^+))$ so that $\psi^*\nu$ and μ agree on L. Choose Cauchy surfaces Σ^- (resp., Σ^+) in $\mathfrak{N}$ containing the Cauchy multi-ball $\psi(\{4t^+/9\} \times B^+)$ (resp., $\psi(\{5t^+/9\} \times B^+)$) and so that $\Sigma^\pm \subset I^\pm_{\mathfrak{N}}(\Sigma^\mp)$.[14] Owing to the split form of the e-metric, $R^\pm \subset I^\pm \times B^\pm$, where $I^\pm = \{t \in \mathbb{R} : |t - t^\pm| < t^+/3\}$. Accordingly, $R^+ \subset (2t^+/3, 4t^+/3) \times B^+ \subset I^+_{\mathfrak{M}}(\{5t^+/9\} \times B^+)$ and hence $\psi(R^+) \subset I^+_{\mathfrak{N}}(\Sigma^+)$; similarly $R^- \subset (-t^+/3, t^+/3) \times B^- \subset I^-_{\mathfrak{M}}(\{4t^+/9\} \times B^+)$ so $\psi(R^-) \subset I^-_{\mathfrak{N}}(\Sigma^-)$. Let F be the closed set formed as the union of $J^\pm_{\mathfrak{N}}(\Sigma^\pm)$ and the closure of $\psi(U)$. Due to the properties of μ and $\Sigma^\pm$, we may choose a smooth function ν on F that vanishes on $J^-_{\mathfrak{N}}(\Sigma^-)$, is identical to unity on $J^+_{\mathfrak{N}}(\Sigma^+)$, and agrees with $\mu \circ \psi^{-1}$ on $\overline{\psi(U)}$. Every point $p \in F$ has a neighbourhood in which ν can be extended to a smooth function taking values in $[0, 1]$—for $p \in J^\pm_{\mathfrak{N}}(\Sigma^\pm)$ this is obvious, while for p in the closure of $\psi(U)$ one may use $\mu \circ \psi^{-1}$. By the smooth Tietze extension theorem (a partition of unity argument) one may obtain an extension of ν in $C^\infty(\mathfrak{N}, [0, 1])$. In particular, ν is a temporal mollifier for $(\psi(R^-), \psi(R^+))$ and $\psi^*\nu$ agrees with μ on U.

Letting $\mathfrak{R}^\pm = \mathfrak{M}|_{R^\pm}$, $\rho^\pm = \iota_{\mathfrak{M};R^\pm}$, $\mathfrak{U} = \mathfrak{M}|_U$, and $\upsilon = \iota_{\mathfrak{M};U}$, parts (i) and (ii) of Lemma 14 apply and give (24). By the timeslice property, $\mathcal{A}(\iota_{\mathfrak{M};D}) = \mathcal{A}(\iota_{\mathfrak{M};R^-}) \circ \vartheta$ for some isomorphism ϑ, because D and R^- are truncated multi-diamonds with a common base (there are Cauchy morphisms from $\mathfrak{M}|_{D\cap R^-}$ to each of $\mathfrak{M}|_D$ and $\mathfrak{M}_{R^-}$). Therefore (25) holds. □

(c) Computation of the 2*-cocycle* The construction of $S \mapsto \zeta(S)$ shows that $\mathcal{A}$ is $\mathcal{S}$-covariant; we now show that the corresponding cocycle (ξ, ϕ) is trivial. Starting with ϕ, let $\alpha \in \mathrm{Aut}(\mathcal{A})$ and consider the diagram

$$\begin{tikzcd}
\mathcal{A}(\mathfrak{M}) \arrow[d, "\alpha_{\mathfrak{M}}"] & \mathcal{A}(\mathfrak{M}^-) \arrow[l] \arrow[r] \arrow[d, "\alpha_{\mathfrak{M}^-}"] & \mathcal{A}(\tilde{\mathfrak{M}}) \arrow[d, "\alpha_{\tilde{\mathfrak{M}}}"] & \mathcal{A}({}^S\mathfrak{M}^+) \arrow[l] \arrow[r] \arrow[d, "\alpha_{{}^S\mathfrak{M}^+}"] & \mathcal{A}({}^S\mathfrak{M}) \arrow[d, "\alpha_{{}^S\mathfrak{M}}"] \\
\mathcal{A}(\mathfrak{M}) & \mathcal{A}(\mathfrak{M}^-) \arrow[l] \arrow[r] & \mathcal{A}(\tilde{\mathfrak{M}}) & \mathcal{A}({}^S\mathfrak{M}^+) \arrow[l] \arrow[r] & \mathcal{A}({}^S\mathfrak{M})
\end{tikzcd}$$

in which unlabelled arrows are isomorphisms arising as images of Cauchy morphisms in (18) and each arrow on the bottom row is identical to the one vertically above it. Each square commutes by naturality of α and one has $\zeta(S)_{\mathfrak{M}}\alpha_{\mathfrak{M}} = \alpha_{{}^S\mathfrak{M}}\zeta(S)_{\mathfrak{M}}$ by definition of $\zeta(S)$. Thus $\phi(S)(\alpha) = \alpha$, for all S and α.

It remains to prove that

$$\zeta_{\mathfrak{M}}(ST) = \zeta_{{}^T\mathfrak{M}}(S)\zeta_{\mathfrak{M}}(T) \qquad (S, T \in \mathcal{S},\ \mathfrak{M} \in \mathsf{FLoc}). \tag{26}$$

Fix $\mathfrak{M} = (\mathcal{M}, e) \in \mathsf{FLoc}$ and choose a Cauchy temporal function $\tau \in C^\infty(\mathcal{M}, \mathbb{R})$ for $\mathfrak{M}$—i.e., $\nabla\tau$ is everywhere e-timelike and future-pointing, and the level sets of τ are smooth spacelike Cauchy surfaces. Also choose disjoint open bounded intervals I^-, I^0, I^+ of $\mathbb{R}$ such that $0 \in I^0$ and $I^\pm \subset \mathbb{R}^\pm$, and define submanifolds $\mathcal{M}^{-/0/+} = \tau^{-1}(I^{-/0/+})$. Finally, choose $\tilde{S}, \tilde{T} \in C^\infty(\mathcal{M}, \mathcal{S})$ so that $\tilde{S} \equiv 1$ on $J^-_{\mathfrak{M}}(\mathcal{M}^0)$ and $\tilde{S} \equiv S$ in $J^+_{\mathfrak{M}}(\mathcal{M}^+)$ while $\tilde{T} \equiv 1$ on $J^-_{\mathfrak{M}}(\mathcal{M}^-)$ and $\tilde{T} \equiv T$ on $J^+_{\mathfrak{M}}(\mathcal{M}^0)$. Then $\tilde{\Lambda}_S = \pi(S)$ and

[14] As $\psi(\{4t^+/9\} \times B^+)$ is a Cauchy multi-ball it lies in a smooth spacelike Cauchy surface Σ^- of $\mathfrak{N}$. Similarly, there exists Σ^+ in the (globally hyperbolic region) $I^+_{\mathfrak{N}}(\Sigma^-)$ containing $\psi(\{5t^+/9\} \times B^+)$. As $\Sigma^\pm$ are Cauchy surfaces, one also has $\Sigma^- \subset I^-_{\mathfrak{N}}(\Sigma^+)$.

$\tilde{\Lambda}_T = \pi(T)$ have the homotopy types of S and T relative to $J^-_{\mathfrak{M}}(\mathcal{M}^0) \cup J^+_{\mathfrak{M}}(\mathcal{M}^+)$ and to $J^-_{\mathfrak{M}}(\mathcal{M}^-) \cup J^+_{\mathfrak{M}}(\mathcal{M}^0)$ respectively. Evidently $\tilde{\Lambda}_S \tilde{\Lambda}_T \in C^\infty(\mathcal{M}, \mathcal{L}_0)$ takes the constant values 1 on $J^-_{\boldsymbol{M}}(\mathcal{M}^-)$, $\pi(T)$ on $\mathcal{M}^0$ and $\pi(ST)$ on $J^+_{\boldsymbol{M}}(\mathcal{M}^+)$, and has the homotopy type of ST relative to $J^-_{\mathfrak{M}}(\mathcal{M}^-) \cup J^+_{\mathfrak{M}}(\mathcal{M}^+)$. Given these choices, we define various spacetimes: $\mathfrak{M}^{-/0/+} = \mathfrak{M}|_{\mathcal{M}^{-/0/+}}$ are slabs of $\mathfrak{M}$ sandwiching the regions where $\tilde{\Lambda}_S$ and $\tilde{\Lambda}_T$ can vary (see Fig. 2), while

$$\tilde{\mathfrak{M}}_T = (\mathcal{M}, \tilde{\Lambda}_T e), \qquad \tilde{\mathfrak{M}}_S = (\mathcal{M}, \tilde{\Lambda}_S \pi(T) e) \qquad \text{and} \quad \tilde{\mathfrak{M}}_{ST} = (\mathcal{M}, \tilde{\Lambda}_S \tilde{\Lambda}_T e)$$

are interpolating spacetimes used in the constructions of $\zeta_{\mathfrak{M}}(T)$, $\zeta_{{}^T\mathfrak{M}}(S)$ and $\zeta_{\mathfrak{M}}(ST)$ respectively. Now consider the following diagram of Cauchy morphisms:

$$\begin{tikzcd}
\mathfrak{M} & \mathfrak{M}^- \arrow[l] \arrow[r] \arrow[d] & \tilde{\mathfrak{M}}_T & & \\
& \tilde{\mathfrak{M}}_{ST} & & {}^T\mathfrak{M}_0 \arrow[ll, dashed] \arrow[ul] \arrow[dl] \arrow[r] & {}^T\mathfrak{M} \\
{}^{ST}\mathfrak{M} & {}^{ST}\mathfrak{M}^+ \arrow[l] \arrow[r] \arrow[u] & \tilde{\mathfrak{M}}_S & &
\end{tikzcd} \tag{27}$$

all of which are induced by inclusion morphisms. (The dashed morphism is well-defined because $\tilde{\Lambda}_S \tilde{\Lambda}_T$ takes the constant value $\pi(T)$ in $\mathcal{M}^0$.) The isomorphism $\zeta_{\mathfrak{M}}(T)$ is obtained by chasing the images under $\mathcal{A}$ of the arrows, left to right, on the upper line from $\mathfrak{M}$ to ${}^T\mathfrak{M}$; $\zeta_{{}^T\mathfrak{M}}(S)$ is obtained by chasing the arrows on the lower line, right to left, from ${}^T\mathfrak{M}$ to ${}^{ST}\mathfrak{M}$, while $\zeta_{\mathfrak{M}}(ST)$ is obtained by chasing from $\mathfrak{M}$ to ${}^{ST}\mathfrak{M}$ along the shortest route. One sees that the identity (26) can be proved by focussing on the central portion of diagram (27) (deleting the external legs) and establishing that the isomorphism from $\mathcal{A}(\mathfrak{M}^-)$ to $\mathcal{A}({}^{ST}\mathfrak{M}^+)$ induced by chasing via ${}^T\mathfrak{M}^0$ is equal to that obtained by chasing vertically downwards. Using the dashed arrow the task splits into two problems, with diagrams

$$\begin{tikzcd}
\mathfrak{M}^- \arrow[rr] \arrow[dd] \arrow[dr, dotted] & & \tilde{\mathfrak{M}}_T \\
& \tilde{\mathfrak{M}} \arrow[ur, dotted] \arrow[dl, dotted] & \\
\tilde{\mathfrak{M}}_{ST} & & {}^T\mathfrak{M}^0 \arrow[ll, dashed] \arrow[uu] \arrow[ul, dotted]
\end{tikzcd} \quad \text{and} \quad
\begin{tikzcd}
\tilde{\mathfrak{M}}_{ST} & & {}^T\mathfrak{M}^0 \arrow[ll, dashed] \arrow[dd] \arrow[dl, dotted] \\
& \hat{\mathfrak{M}} \arrow[ul, dotted] \arrow[dr, dotted] & \\
{}^{ST}\mathfrak{M}^+ \arrow[uu] \arrow[rr] \arrow[ur, dotted] & & \tilde{\mathfrak{M}}_S
\end{tikzcd}$$

where again we must show the equivalence of the chase around the right-hand portions to that obtained by passing vertically down from $\mathfrak{M}^-$ or $\tilde{\mathfrak{M}}_{ST}$ respectively. In these diagrams, the solid and dashed morphisms are those in the previous diagram, $\tilde{\mathfrak{M}} = \tilde{\mathfrak{M}}_T|_{J^-_{\mathfrak{M}}(\mathcal{M}^0)}$, $\hat{\mathfrak{M}} = \tilde{\mathfrak{M}}_{ST}|_{J^+_{\mathfrak{M}}(\mathcal{M}^0)}$, and the dotted morphisms are defined by inclusion maps. Every small triangle in these diagrams is a commuting triangle of Cauchy morphisms induced by an inclusion. Taking images under $\mathcal{A}$, every small triangle is a commuting triangle of isomorphisms and therefore the isomorphisms induced by chasing along the right-hand portions of the diagrams coincide with the isomorphism induced by the left-hand vertical line. This concludes the proof of (26) and hence of Theorem 11.

4.3. Minkowski space. Define n-dimensional Minkowski space to be the object $\mathfrak{M}_0 = (\mathbb{R}^n, (dX^\mu)_{\mu=0}^{n-1})$ of FLoc, where $X^\mu : \mathbb{R}^n \to \mathbb{R}$ are the coordinate functions $X^\mu(x^0, \ldots, x^{n-1}) = x^\mu$. The corresponding object $\boldsymbol{M}_0 := \mathfrak{F}_{\mathsf{L}}(\mathfrak{M}_0)$ of Loc has the restricted Poincaré group as its group of automorphisms. In FLoc, however, Lorentz symmetry is broken by the choice of frame and $\mathcal{M}_0$ only admits spacetime translations as automorphisms. Instead, Lorentz transformations map between distinct objects: each $\Lambda \in \mathcal{L}_0$ induces an active Lorentz transformation $\psi_\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ by matrix multiplication, $X^\mu \circ \Lambda = \Lambda^\mu{}_\nu X^\nu$; as $\psi_\Lambda^* dX^\mu = \Lambda^\mu{}_\nu dX^\nu$, ψ_Λ induces an morphism $\psi_\Lambda : \mathcal{M}_0 \to {}^{\Lambda^{-1}}\mathcal{M}_0$ in FLoc. Given a second $\Lambda' \in \mathcal{L}_0$, the morphism ${}^{\Lambda^{-1}}\psi_{\Lambda'} : {}^{\Lambda^{-1}}\mathcal{M}_0 \to {}^{(\Lambda'\Lambda)^{-1}}\mathcal{M}_0$ has underlying map Λ', and therefore ${}^{\Lambda^{-1}}\psi_{\Lambda'} \circ \psi_\Lambda$ has underlying map $\Lambda'\Lambda$, giving the equality of morphisms

$$\psi_{\Lambda'\Lambda} = {}^{\Lambda^{-1}}\psi_{\Lambda'} \circ \psi_\Lambda. \tag{28}$$

As ψ_Λ has inverse ${}^{\Lambda^{-1}}\psi_{\Lambda^{-1}}$ it is therefore an isomorphism in FLoc. Of course, $\mathcal{F}_{\mathsf{L}}(\psi_\Lambda)$ is simply the Lorentz transformation Λ as an automorphism of $\boldsymbol{M}_0$. To economize on notation we also write ψ_S for $\psi_{\pi(S)}$ if $S \in \mathcal{S}$.

Whereas a functor on Loc automatically represents Lorentz transformations in the automorphism group of $\mathcal{A}(\boldsymbol{M}_0)$, the same is not true of functors on FLoc. This is remedied precisely by $\mathcal{S}$-covariance: for each $S \in \mathcal{S}$, we may define

$$\Xi(S) = \mathcal{A}({}^S\psi_S) \circ \zeta(S)_{\mathcal{M}_0} = \zeta(S)_{S^{-1}\mathcal{M}_0} \circ \mathcal{A}(\psi_S), \tag{29}$$

an automorphism of $\mathcal{A}(\mathcal{M}_0)$ with some important properties. First, it is clear that $\Xi(1) = \mathrm{id}_{\mathcal{A}(\mathcal{M}_0)}$ and more generally that, if $S \in \ker \pi$ covers the identity Lorentz transformation, then $\Xi(S) = \zeta(S)_{\mathcal{M}_0}$. For example, in $n \geq 4$ spacetime dimensions, $\Xi(-1) = \zeta(-1)_{\mathcal{M}_0}$. Second, note that

$$\begin{aligned}
\Xi(S')\Xi(S) &= \mathcal{A}({}^{S'}\psi_{S'})\zeta(S')_{\mathcal{M}_0}\mathcal{A}({}^S\psi_S)\zeta(S)_{\mathcal{M}_0} \\
&= \mathcal{A}({}^{S'}\psi_{S'})\mathcal{A}({}^S\psi_S)\zeta(S')_{S\mathcal{M}_0}\zeta(S)_{\mathcal{M}_0} \\
&= \mathcal{A}({}^{S'S}\psi_{S'S})\zeta(S'S)_{\mathcal{M}_0} = \Xi(S'S),
\end{aligned} \tag{30}$$

where, in the penultimate step, we use the fact that ζ has a trivial cocycle, and also the identity (28). Third, the action on fields is

$$\begin{aligned}
\Xi(S)\Phi_{\mathcal{M}_0}(f) &= \mathcal{A}({}^S\psi_S)\zeta(S)_{\mathcal{M}_0}\Phi_{\mathcal{M}_0}(f) = \mathcal{A}({}^S\psi_S)(S * \Phi)_{S\mathcal{M}_0}(f) \\
&= (S * \Phi)_{\mathcal{M}_0}(\pi(S)_* f)
\end{aligned}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$ ($\mathcal{D}$ is $\mathcal{S}$-covariant with a trivial implementation).

Fourth, given a state ω_0 on $\mathcal{A}(\mathcal{M}_0)$ that is invariant under these automorphisms, i.e., $\omega_0 \circ \Xi(S) = \omega_0$ for all S, the corresponding GNS representation $(\mathcal{H}_0, \mathcal{D}_0, \pi_0, \Omega_0)$ will carry a unitary implementation of the $\Xi(S)$, so that

$$\pi_0(\Xi(S)A) = U(S)\pi_0(A)U(S)^{-1}, \qquad U(S)\Omega_0 = \Omega_0$$

for all $S \in \mathcal{S}$, recovering the standard transformation laws of fields in Minkowski QFT. The use of FLoc has distinguished two aspects of the Lorentz transformation: the active transformation of points and algebra elements under $\mathcal{A}(\psi_S)$, and the passive relabelling of field multiplets arising from $\mathcal{S}$-covariance.

5. Conclusion

We have given a general analysis of G-covariance of locally covariant theories in terms of nonabelian cohomology. Among the general features uncovered are the existence of an associated canonical cohomology class, and the structure of field multiplets. As well as discussing rigid scale covariance, we have established a no-go theorem on mixing of internal and Lorentz symmetries analogous to the Coleman–Mandula theorem. Our approach here is completely new and makes no use of Minkowski spacetime structures. This gives a new perspective on results of this type and further demonstrates the utility of relative Cauchy evolution.

Directions in which this work could be extended include the following. First one could study smooth G-covariance using, e.g., the smooth nonabelian cohomology of [31]. Topologies on $\mathrm{Aut}(\mathcal{A})$ and $\mathrm{Fld}(\mathcal{A})$ can be given in terms of suitable state spaces [15, Sect. 3.2]. Second, the proof of Theorem 11 would apply to other rigid group actions that can be achieved by smooth deformation (e.g., the conformal group). Finally, an obvious question is whether an analogue of the Haag–Łopuszański–Sohnius theorem [22] can be proved for theories on a suitable category of supermanifolds, perhaps using the enriched category methods of [23].

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